MATHEMATICS FOR PROSPECTIVE MEMBERS OF AIRCRAFT CREWS.

AN APPEAL TO TEACHERS.

THERE is an urgent need for the expansion of the R.A.F. Many young men who are physically and temperamentally fit for the service are unfortunately lacking in the necessary knowledge of elementary mathematics. For some time the Regional Committees for Adult Education have been giving assistance to such men, but the time has now come when this help must be greatly increased and extended to men of lower initial attainments. Cases differ so much that only very small classes are possible, or individual tuition may be necessary. Usually the only time that the men are free is in the evening.

A syllabus of the minimum course is appended :

>   Fractions—problems ;
>   Conversions from Metric to British units ;
>   Averages, Ratio and Proportion, Percentages ;
>   Relation between the sides of a right angle triangle and square roots (proof not required) ;
>   Transposition of equations ;
>   Problems using simple equations ;
>   Graphs (not algebraic) ;
>   Triangle of velocities.

Will teachers who are willing to help in this vital national service please communicate with a Regional Committee (address c/o the nearest University or University College) or the local Director of Education ?

A

ELEMENTARY INEQUALITIES.

By T. Arnold Brown.

1. The article by Mrs. Linfoot on the " Teaching of Elementary Inequalities " published in the issue of the *Mathematical Gazette* for July, 1940, raises a matter of considerable interest and importance.

It is unhappily the case that the elementary treatment of inequalities remains a weak point in English mathematical teaching. While no Higher School Certificate examination paper in Algebra would be considered complete without the inclusion of questions on identities and equations, yet direct questions on inequalities and inequations occur comparatively rarely and, indeed, even when such questions are included in University Entrance Scholarship examinations, they tend to be ignored by the majority of candidates. While the subject receives on the whole adequate treatment in the standard textbooks, it is generally relegated to a late chapter and is liable to be crowded out of the school curriculum ; there is no explicit reference to it in the Report of the Mathematical Association on the Teaching of Algebra in Schools. The result is that the university student may be invited to discuss the convergence of an infinite series before he is competent to assign an upper limit to the numerical value of the remainder.

Professor Richardson has recently made a plea for the introduction into elementary teaching of the ideas and methods of modern abstract algebra.* Now the notion of an *ordering relation* is well exemplified by inequalities and some acquaintance with them would appear to be an almost essential preliminary to any attempt to give effect to such a proposal.

If any evidence were required to establish the fundamental importance of inequalities in many branches of higher mathematics, it would be provided by a perusal of the volume on the subject written by G. H. Hardy, J. E. Littlewood and G. Pólya.†

The suggestion made by Mrs. Linfoot is therefore very welcome. The introduction by way of geometry, or otherwise, of the basic ideas of the theory at the School Certificate stage would pave the way for the analytical development which should follow as soon as the pupil is in a position to appreciate the significance of the associative, commutative and distributive laws of ordinary algebra.

Inequalities, of course, depend just as closely as identities upon these laws, but it is precisely this point which even university students frequently fail to grasp. Their obvious reluctance to deal with inequalities appears to be engendered by the belief that they depend upon an unfamiliar and vaguely defined order of ideas. Nevertheless, if properly presented, they serve to illustrate in a valuable way the logical basis of general algebraic processes while

* *Mathematical Gazette*, February, 1940, 15.

† *Inequalities* (Cambridge, 1934).

at the same time furnishing in many cases a clearer picture of arithmetical relationship than that provided by mere identities.

In what follows an attempt is made to indicate briefly how the subject may be approached, to exhibit its connection with other branches of elementary algebra and to point out one or two of the more interesting and attractive applications.

2. The field in which inequalities operate is the continuum of positive and negative real numbers, but little is sacrificed in the way of generality if we restrict ourselves to commensurable numbers in the first instance. We observe that sums, products and quotients of positive numbers are all positive, and we define the inequalities $a > b$ or $b < a$ as equivalent to the statement that the difference $a - b$ is positive.

The primary propositions relate to the algebraic operations of addition, multiplication and exponentiation, and are as follows :

If $a > b$, then

(2·1) $$a \pm c > b \pm c,$$
(2·2) $$ca > cb, \ if \ c > 0,$$
$$ca < cb, \ if \ c < 0.$$

These depend directly upon the associative and distributive laws respectively. Thus

$$(a \pm c) - (b \pm c) = a - b,$$
$$ca - cb = c(a - b).$$

The operations of addition and multiplication (or division) also enable us to combine two or more inequalities.

If $a > b$ and $c > d$, then

(2·3) $$a + c > b + d.$$

If a, b, c, d be all positive, and if $a > b$ and $c > d$, then

(2·4) $$ac > bd \ and$$
(2·5) $$\frac{a}{d} > \frac{b}{c}.$$

The first two depend upon the obvious identities

$$(a + c) - (b + d) = (a - b) + (c - d),$$
$$ac - bd = c(a - b) + b(c - d),$$

while (2·5) follows from (2·4) on division by the positive product cd.

The results expressed by (2·3) and (2·4) may be extended by a repetition of the argument to the sum or product of any finite number of inequalities.

We now consider *exponentiation*.

If (i) a, b be positive, (ii) $a > b$, (iii) r, s be positive integers, and if $a^{r/s}$ denote the positive sth root of a^r, then

(2·6) $$a^{r/s} > b^{r/s} \ and$$
(2·7) $$a^{-r/s} < b^{-r/s}.$$

We have

$$\frac{a^{r/s} - b^{r/s}}{a - b} = \frac{c^r - d^r}{c^s - d^s}, \text{ where } a = c^s, \ b = d^s,$$

(2·8)
$$= \frac{c^{r-1} + c^{r-2}d + \ldots + d^{r-1}}{c^{s-1} + c^{s-2}d + \ldots + d^{s-1}}$$

$$> 0,$$

since all the terms are positive.

If we multiply by $a - b$, which is positive, we obtain (2·6) ; (2·7) is derived from (2·6) on multiplying by the positive product $a^{-r/s}b^{-r/s}$.

Only a superficial use is here made of identity (2·8), which involves all three fundamental laws and goes quite deep. We shall find scope for a fuller application in the next section.

We may illustrate the application of these elementary theorems by means of the following *example* :

Given that $0 \leqslant x < 1$ and $0 \leqslant \theta \leqslant 1$, prove that, for every positive integer n,

$$\left(\frac{x - \theta x}{1 - \theta x}\right)^n < 1.$$

We have at once by (2·1)

$$x - \theta x < 1 - \theta x.$$

Now, if we multiply the inequality $\theta \leqslant 1$ by x, we obtain

$$\theta x \leqslant x \quad \text{by (2·2)}$$

and therefore $\quad\quad\quad x - \theta x \geqslant 0 \quad \text{by (2·1)}.$

Again, if we multiply the two inequalities

$$x < 1 \quad \text{and} \quad \theta \leqslant 1,$$

we find $\quad\quad\quad\quad \theta x < 1 \quad \text{by (2·4)}.$

Therefore $\quad\quad\quad 1 - \theta x > 0 \quad \text{by (2·1)}.$

Combining all three results, we have

$$0 \leqslant \frac{x - \theta x}{1 - \theta x} < 1$$

and the required result follows from (2·6).

This example finds an interesting application in the discussion of the convergence of the Maclaurin series for $\log (1 - x)$ in the range $0 \leqslant x < 1$.

3. It is but a short step from these extremely elementary considerations to the most celebrated of all theorems on inequalities, namely, the Theorem of the Means.

Suppose that c and d are positive numbers such that $c > d$ and that n is a positive integer. Then, in virtue of (2·2), we have

$$c^{n-1} > c^{n-2}d > c^{n-3}d^2 > \ldots > d^{n-1};$$

Hence from (2·8), (2·5) and (2·2) it follows that

$$\frac{c^r - d^r}{c^s - d^s} < \frac{rc^{r-1}}{sd^{s-1}} < \frac{rc^r}{sd^s}.$$

Similarly, if $c < d$, we have

$$\frac{c^r - d^r}{c^s - d^s} > \frac{rc^r}{sd^s}.$$

Clearing of fractions we find in either case by (2·2) and (2·6) that

$$rc^r(c^s - d^s) > sd^s(c^r - d^r).$$

Hence, provided only that c and d are unequal, we have

(3·1) $$rc^{r+s} + sd^{r+s} > (r+s)\,c^r d^s.$$

Clearly the limiting case of equality occurs if, and only if, $c = d$.
Now write
$$b_1 = c^{r+s}, \qquad b_2 = d^{r+s},$$
$$q_1 = \frac{r}{r+s}, \qquad q_2 = \frac{s}{r+s},$$

so that q_1, $q_2 > 0$, $q_1 + q_2 = 1$. Then (3·1) becomes

(3·2) $$q_1 b_1 + q_2 b_2 > b_1^{q_1} b_2^{q_2}.$$

This is the simplest form of the theorem with " weighted means ".
Putting $q_1 = q_2 = \frac{1}{2}$, and replacing b by a, we obtain the most elementary form of the theorem, namely,

(3·3) $$\tfrac{1}{2}(a_1 + a_2) > \sqrt{(a_1 a_2)}.$$

It is now a simple matter to establish the general form of this result by the ordinary process of direct induction.
Suppose that the inequality

(3·4) $$a_1 + a_2 + \ldots + a_n > n(a_1 a_2 \ldots a_n)^{1/n}$$

is true for some particular integer n.
Then we have

$$\frac{a_1 + a_2 + \ldots + a_n + a_{n+1}}{n+1} > \frac{n}{n+1}(a_1 a_2 \ldots a_n)^{1/n} + \frac{a_{n+1}}{n+1}$$

$$> \{(a_1 a_2 \ldots a_n)^{1/n}\}^{\frac{n}{n+1}} \{a_{n+1}\}^{\frac{1}{n+1}},$$

by (3·2),

$$= (a_1 a_2 \ldots a_{n+1})^{\frac{1}{n+1}}$$

and the inequality is therefore true for the next integer $(n+1)$.
But by (3·3) it is true for $n = 2$ and hence it is true generally.
It will be observed that the inequality reduces to an obvious identity if, and only if, all the a are equal.
The cases $n = 3$ and $n = 4$ are easily obtained directly. Thus, by a well-known identity, we have

$$a_1^3 + a_2^3 + a_3^3 - 3a_1 a_2 a_3 = \tfrac{1}{2}(a_1 + a_2 + a_3)\,\Sigma(a_2 - a_3)^2,$$

which is strictly positive provided that the a are real, positive and not all equal.

Again, by a repeated appeal to (3·3), we have

$$a_1{}^4 + a_2{}^4 + a_3{}^4 + a_4{}^4 > 2(a_1 a_2)^2 + 2(a_3 a_4)^2$$
$$> 4(a_1 a_2 a_3 a_4).$$

It is indeed evident that the case $n = 2^m$ may similarly be established somewhat after the manner of the famous " condensation test " of Cauchy. But the passage from $n = 2^m$ to general n is by no means obvious and involves some *tour de force* such as " backward induction ",[*] with which the student is in all probability quite unfamiliar. The reason for this difficulty is not far to seek ; (3·3) is a much less significant statement than (3·2) and provides, therefore, a rather weak starting-point for the development of the argument. But ordinary direct induction is perfectly feasible provided that weighted means are employed in the first instance at least.

The general result with weighted means, namely,

$$(3·5) \qquad \Sigma qb > \Pi b^q,$$

may be derived from (3·4) by supposing the a subdivided into suitable groups of equal numbers or it may be deduced directly from (3·2) with the aid of (2·6).

Thus, if $q_1 + q_2 + q_3 = 1$, we have

$$q_1 b_1 + q_2 b_2 + q_3 b_3 = (q_1 + q_2)\frac{q_1 b_1 + q_2 b_2}{q_1 + q_2} + q_3 b_3$$
$$> \left(\frac{q_1 b_1 + q_2 b_2}{q_1 + q_2}\right)^{q_1 + q_2} b_3{}^{q_3}$$
$$> b_1{}^{q_1} b_2{}^{q_2} b_3{}^{q_3},$$

and so generally.

Many proofs [†] of these theorems have been constructed, but the method outlined above has the advantage of depending upon a single order of elementary ideas and represents probably the best line of attack for a first approach to the subject.

4. It is appropriate to consider at this stage a more general type of inequality due to Jensen,[‡] which includes the Theorem of the Means as a particular case and admits of very similar treatment.

Jensen's Inequality is rather less elementary, however, for it involves the notion of a *convex function*.

A real function $f(x)$ defined in the open interval (a, b) is said to be *convex*,[§] if

---

[*] Hardy, and others, *loc. cit.*, 20.

[†] Full references will be found in **Hardy**, *loc. cit.*, 16 *et seq.*

[‡] J. L. W. V. Jensen, Sur les fonctions convexes et les inégalités entre les valeurs moyennes, *Acta Math.* 30 (1906), 175-193.

[§] Cf. E. **Landau**, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie* (Berlin, 1916), 77.

(4·1)
$$f(x) < \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$$

for
$$a < x_1 < x < x_2 < b.$$

Geometrically this means that the graph of $y = f(x)$ between x_1 and x_2 always lies below the chord joining the points $\{x_1, f(x_1)\}$ and $\{x_2, f(x_2)\}$.

If we write
$$q_1 = \frac{x_2 - x}{x_2 - x_1}, \quad q_2 = \frac{x - x_1}{x_2 - x_1},$$

then we have
$$q_1 > 0, \quad q_2 > 0, \quad q_1 + q_2 = 1,$$

and
$$x = q_1 x_1 + q_2 x_2,$$

so that condition (4·1) becomes

(4·2)
$$f(q_1 x_1 + q_2 x_2) < q_1 f(x_1) + q_2 f(x_2).$$

It is evident, by the section formula of geometry, that $q_2 : q_1$ is the ratio in which the point on the chord whose abscissa is x divides the join of $\{x_1, f(x_1)\}$ and $\{x_2, f(x_2)\}$.

Again, if we write $q_1 = q_2 = \frac{1}{2}$, the *strong* condition (4·1) is replaced by the *weak* * one

(4·3)
$$f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2}.$$

This inequality may be generalized as before by means of ordinary induction and a single appeal to (4·2). Thus suppose that

(4·4)
$$f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) < \frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n}$$

for some particular $n \geqslant 2$, where
$$a < x_1 < x_2 \ldots < x_n < b.$$

Then, if $x_n < x_{n+1} < b$, we have
$$f\left(\frac{x_1 + x_2 + \ldots + x_{n+1}}{n+1}\right) = f\left(\frac{n}{n+1} \frac{x_1 + \ldots + x_n}{n} + \frac{x_{n+1}}{n+1}\right)$$
$$< \frac{n}{n+1} f\left(\frac{x_1 + \ldots + x_n}{n}\right) + \frac{1}{n+1} f(x_{n+1}), \quad \text{by (4·2)}$$
$$< \frac{n}{n+1} \frac{f(x_1) + \ldots + f(x_n)}{n} + \frac{1}{n+1} f(x_{n+1}), \quad \text{by (4·4)},$$
$$= \frac{f(x_1) + f(x_2) + \ldots + f(x_{n+1})}{n+1},$$

and so (4·4) is true for the next integer $n + 1$ and therefore generally.

The situation is thus precisely similar to that which we encountered in dealing with the Means. By simple repetition of the argu-

---

* If $f(x)$ is *continuous* (4·2) and (4·3) are equivalent. Cf. **Hardy**, *loc. cit.*, 73.

ment the strong condition (4·4) may be extended at once to the general case, namely,

(4·5) $f(\Sigma qx) < \Sigma q f(x).$

The weak condition (4·3) will also serve, though less conveniently, as a starting-point first to establish (4·4) for $n = 2^m$, then for general n and, finally, by supposing the x replaced by suitable sets of equal numbers to yield (4·5).

These theorems find interesting applications in many branches of modern function-theory and particularly in regard to the Maximum Modulus Theorem[*] of analysis.

It may be observed that, if in Jensen's Inequality (4·4) or (4·5) we put $f(x) = -\log x$, we obtain the Theorem of the Means.

5. Inequalities which involve one or more variables $x, y, \ldots$ are sometimes called *inequations*. These stand in very much the same relation to inequalities as do equations to identities and, if confusion of thought is to be avoided, it is just as necessary to preserve the distinction in the one case as in the other. Once the student has acquired some degree of confidence in manipulating inequalities, but not before, he will be qualified to proceed to the consideration of inequations. All that is necessary in the first instance is to replace the numbers $a, b, \ldots$, previously employed, by integral expressions in x, say, $P, Q, \ldots$ and to restate in appropriate fashion the various primary propositions of Section 2. Thus, for example, the analogue of theorem (2·1) will be as follows :

Every x which satisfies

(5·1) $P > Q$
will satisfy
(5·2) $P + R > Q + R$
and conversely.

The inequations (5·1) and (5·2) are in fact *equivalent*.

The restatement of the remaining theorems may be left to the reader, but it may be observed that the analogues of (2·3), (2·4) and (2·5) will not be required unless systems of simultaneous inequations are being considered and that we may generally restrict ourselves to the index 2 in applications of (2·6).

The most important inequation in elementary work is naturally the quadratic inequation and the relevant theorem may be immediately derived from the foregoing principles in the following form :

The quadratic expression

$$ax^2 + bx + c$$

has the same sign as a except when the roots of the expression are real and x lies between them.

It will not be necessary to develop the matter in detail here, and an example or two will suffice to illustrate the general method of treatment.

[*] Cf. E. C. Titchmarsh, *The Theory of Functions* (Oxford, 1932), 173.

Example 1. Solve the inequation

$$\frac{x-1}{x-2} < \frac{4-x}{7}.$$

We may multiply by the factor $7(x-2)^2$, which is strictly positive, since $x=2$ is clearly not a relevant value of x; we thus obtain the equivalent inequation

$$7(x-1)(x-2) < (4-x)(x-2)^2,$$

or $$(x-2)(x^2+x+1) < 0,$$

or $$x-2 < 0,$$

on dividing by x^2+x+1, which is strictly positive for all x.

Hence the solution is $x < 2$.

Geometrically this means that the straight line $y=(4-x)/7$ lies evenly between the two branches of the rectangular hyperbola $y=(x-1)/(x-2)$ without meeting either.

Example 2. Find the necessary and sufficient condition that the roots of the two equations

$$x^2 + px + q = 0 \quad \text{and} \quad x^2 + p'x + q' = 0$$

should " interlace ".

For this to happen the roots must all be real. If those of the first equation be denoted by α and β, then a sufficient condition is evidently

(5·3) $$(\alpha^2 + p'\alpha + q')(\beta^2 + p'\beta + q') < 0,$$

which reduces to

(5·4) $$(q-q')^2 + (p-p')(pq' - p'q) < 0$$

on utilizing the relations between the roots and the coefficients of the first equation.

The steps of the argument are reversible and starting with (5·4) we may derive either (5·3) or the corresponding inequality which involves the roots of the second equation and the coefficients of the first. Neither is consistent with the occurrence of a pair of conjugate complex roots. Hence all the roots are real and they interlace. The condition is therefore necessary as well as sufficient.

The limiting case of (5·4) when equality occurs corresponds to the problem of eliminating x between the two equations, and the example sheds much light upon the significance of this process. The inequality (5·4) possesses, moreover, considerable intrinsic interest, for, when treated as an inequation, it serves to determine the range of values in which each of the four real numbers p, q, p', q' must lie, when the other three are given, in order that interlacing may take place. Thus it may be shown that q' must lie within the interval specified by

$$q - (p-p')\{p \pm (p^2 - 4q)^{\frac{1}{2}}\}/2,$$

while p' must lie within or without the interval

$$\{p(q+q')\pm(q-q')(p^2-4q)^{\frac{1}{2}}\}/2q$$

according as q is positive or negative. The inequality is thus intrinsically inconsistent with either or both of the inequalities $p^2 < 4q$ and $p'^2 < 4q'$. We have here a very typical case of elementary algebraic elaboration.

In the solution of inequations in general we are concerned with sets of numbers which are everywhere dense in a region of one or more dimensions and their study consequently provides, as Mrs. Linfoot points out, a valuable preparation for the eventual discussion of limits and functions.

6. Equations are, strictly speaking, limiting cases of inequations and, logically at least, the treatment of the latter should precede that of the former. Since this order of development will seldom be attained in practice it is all the more important to display to advantage the essential uniformity in the basic principles which govern operations on inequalities, inequations and equations alike and explicitly to utilize, with minor modifications only, the same set of primary propositions throughout. The student will thus more readily realize that, in the diversity of algebraic manipulations, there is a unity dictated ultimately by the fundamental laws of the subject. Moreover, it is not possible, particularly in applications to mechanics, to maintain a strict separation between the various topics of elementary algebra. It may be necessary to handle simultaneous systems in which inequations occur side-by-side with equations. Such is the case, for example, when we investigate the condition that a heavy particle moving on a smooth surface of revolution with axis vertical should maintain contact with the surface throughout the motion. A similar situation arises in statics whenever multiple systems subject to mutual friction are under consideration.

So far as ordinary equations in one variable are concerned, we require only the analogues of theorems (2·1), (2·2) and (2·6) in the following forms :

(6·1)   *The equations $P = Q$ and $P + R = Q + R$ are equivalent.*

(6·2)   *The roots of $P = Q$ satisfy $RP = RQ$; the roots of $RP = RQ$ are the roots of $P = Q$ together with those of $R = 0$.*

(6·3)   *The roots of $P = Q$ satisfy $P^2 = Q^2$; the roots of $P^2 = Q^2$ are the roots of $P = Q$ together with those of $P = -Q$.*

The analogues of (2·3) and (2·4) are available for the treatment of systems of simultaneous equations, if desired.

The results expressed by (6·2) and (6·3) deserve special consideration since they are connected with the question of extraneous roots, and this is a matter which seems to give rise to much doubt and confusion.

We may illustrate by reference to two standard cases.

Let P, Q, R denote linear expressions in x and consider the equation

$$\sqrt{P} + \sqrt{Q} = \sqrt{R}.$$

It is usually rationalized by successive squaring so as to yield

$$\Sigma P^2 \quad 2\Sigma QR = 0.$$

This quadratic equation has two roots, either or both of which may be extraneous.

As soon as it is understood that the above process is equivalent to multiplying the given equation by the rationalizing factor

$$(\sqrt{P} - \sqrt{Q} - \sqrt{R})(\sqrt{P} - \sqrt{Q} + \sqrt{R})(\sqrt{P} + \sqrt{Q} + \sqrt{R}),$$

the mystery of the extraneous roots is resolved by an appeal to (6·2). They are the roots of one or other of the equations obtained by equating to zero each of the separate factors in the above expression. This point of view has an incidental advantage in showing that an equation does not necessarily possess any finite root (real or complex).

As a last example we may consider the well-known equation

$$a \cos x + b \sin x = c.$$

The average student does not readily appreciate the value of the special artifices which are introduced to deal with this case. His natural inclination is to proceed by squaring so as to obtain the equation

$$(a \cos x - c)^2 = b^2 \sin^2 x = b^2(1 - \cos^2 x)$$

and to solve the resulting quadratic in $\cos x$. The only reasonable way to convince him that this is bad practice is to show by an appeal to (6·3) that the roots of the equation

$$a \cos x - b \sin x = c$$

will be introduced as extraneous roots by this process. He will then be glad to avail himself of any method which will save him the trouble of separating the relevant from the irrelevant roots.

T. A. B.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

THE CISSOID OF DIOCLES.

By J. P. McCarthy.

1. THE construction, presumably used by Diocles,* who flourished in the second century B.C., for the cissoid is as follows.
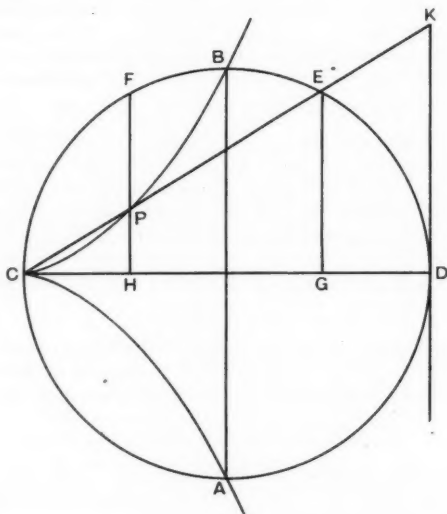
FIG. 1.

Let AB, CD be perpendicular diameters of a circle. Let E, F be points on the quadrants BD, BC such that the arcs BE, BF are equal. Draw EG, FH perpendicular to CD. Join CE and let P be the point of intersection of CE and FH. The cissoid is the locus of P corresponding to points E, F on the quadrants BD, BC such that the arcs BE, BF are equal.

2. If CPE is produced to meet the tangent at D to the circle in K, then it is clear that $CH = GD$ and consequently $CP = EK$. This gives the following alternative construction of the curve.

Let the diameter of the circle be $2a$. Take CD as axis of x and the tangent CY as axis of y. Draw the chord CE to cut the tangent DK at K and make $CP = EK$. Then the locus of P is the cissoid.

3. To obtain the equation to the curve, using polar coordinates, let P be the point (r, θ), CX being the initial line. Then

$$CE = CD \cos \theta = 2a \cos \theta$$

and

$$CK = CD \sec \theta = 2a \sec \theta.$$

* Heath, *Greek Mathematics*, I, p. 264 (Oxford).

Hence

$$r = CP = EK = CK - CE$$
$$= 2a(\sec\theta - \cos\theta),$$

that is,

$$r = \frac{2a\sin^2\theta}{\cos\theta}. \quad\text{.....................................(i)}$$

The equation when expressed in Cartesian coordinates takes the form

$$y^2 = x^3/(2a - x). \quad\text{.............................(ii)}$$
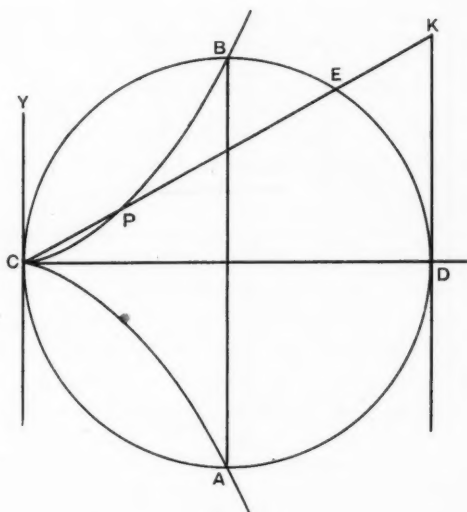
FIG. 2.

In either form the equation shows that the curve is symmetrical about the axis of x. Equation (ii) shows that the line $x = 2a$, that is, the tangent at D, is an asymptote. In addition, there is a cusp at the origin C. The curve passes through the points $(a, \pm a)$, that is, through A and B, and if y is real, x lies between 0 and $2a$.

4. As is well known, the historical interest in the curve lies in its application to the duplication of the cube, for it was to solve this problem that Diocles invented the curve. In the fifth century B.C. Hippocrates of Chios had shown that the duplication problem was equivalent to that of finding two mean proportionals between two given lines, one twice the length of the other, and, so Proclus says, thereafter attention was directed exclusively to the equivalent

problem. For, if x and y are mean proportionals between l and $2l$ we have

$$l : x = x : y = y : 2l = \sqrt[3]{\frac{1}{2}},$$

whence $$x^3 = 2l^3.$$

Hence the cube whose edge is x is double that whose edge is l.

5. By means of the cissoid we can construct the two mean proportionals between two lines, one of which is twice the other.
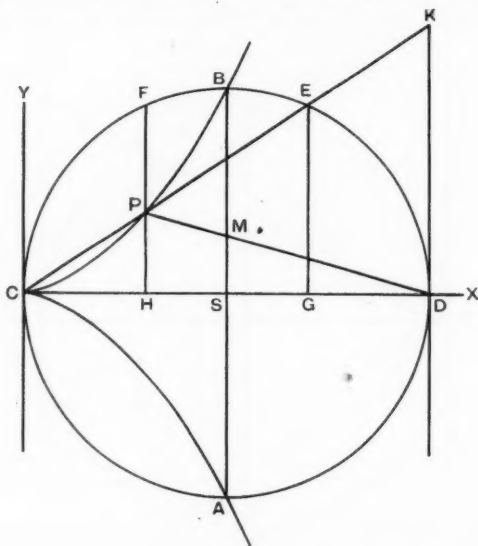
FIG. 3.

Let S be the centre of the circle, $2a$ the diameter, and M the middle point of SB. Let DM meet the cissoid at P. Draw the secant $CPEK$, the ordinates FPH and EG.

Since $$CP = EK,$$

we have $$CH = GD$$

and $$HS = SG.$$

If $$CH = x,\ HF = y,\ DH = 2l,$$

then $$DH : HP = 2l : HP$$

$$= DS : SM$$

$$= SB : SM$$

$$= 2. \quad ....(iii)$$

Therefore, $$HP = l. \quad ....(iv)$$

Now, the triangles CHP, CGE are similar ;

the triangles CGE, DHF are congruent ;

the triangles DHF, FHC are similar.

Hence $\qquad DH : HF = FH : HC = CH : HP$, ...(v)

that is, FH and HC are the mean proportionals required.

Hence $\qquad 2l : y = y : x = x : l = \sqrt[3]{2}$, ...(vi)

and $\qquad\qquad\qquad x^3 = 2l^3$; ....(vii)

or, the cube whose edge is CH is double that whose edge is HP.

6. In general,* we can find two mean proportionals between two given lengths a and b. For, assuming that a is greater than b, if we make (Fig. 3)

$$SB : SM = a : b,$$

we have, from (iii),

$$DH : HP = a : b,$$

and we have to find straight lines which bear the same ratio to DH, HF, HC and HP as the length a bears to DH. The first and last of these will be a, b respectively, and the others will be the mean proportionals required.

7. We can obtain by means of the cissoid a geometrical construction for the cube root of any number n.

If $SB : SM = n$, we have, from (iii), $DH : HP = n$,

and $\qquad 2l : y = y : x = x : 2l/n = \sqrt[3]{n}$, from (v),

whence $\qquad\qquad y : 2l = 1 : \sqrt[3]{n}$.

Hence the cube root required is the fourth proportional to lengths equal to y, $2l$ and unity.

8. For the purpose for which it was invented, only that part of the cissoid lying within the circle was required. It appears to have been first noticed by Roberval in 1640 that the curve extends to infinity in both directions. $\qquad\qquad$ J. P. McC.

EUREKA.

THE latest issue of our sprightly contemporary, *Eureka*, is now on sale. Owing to the increase in cost of printing, etc., the price has been raised to 8d., postage $2\frac{1}{2}$d. Copies can be obtained on application to :

$\qquad$ THE BUSINESS MANAGER,

$\qquad\qquad$ *Eureka*,

$\qquad\qquad\qquad$ c/o Mathematical Faculty Library,

$\qquad\qquad\qquad\qquad$ New Museums,

$\qquad\qquad\qquad\qquad\qquad$ CAMBRIDGE.

$\qquad\qquad$ * Heath, *loc. cit.*

RUG DESIGNING.

By N. M. Gibbins.

The usual method of making rugs with cut wool on canvas is to hook a piece of wool into each hole. This process causes the pieces to become very crowded ; and, besides being slow in showing results, it is laborious and costly. It is usual, moreover, to follow a pattern drawn on the canvas, and the pattern is seldom one to appeal to a mathematician, Victorian in conception as it so often is with its display of flowers and fruits.

The object of this article is to exhibit the far-reaching consequences of filling up every other hole instead of every hole. An immediate consequence is that the work proceeds with less than half the labour and more than half the speed, because there is plenty of freedom for the hook. Experience has shown that no glimpses of canvas appear once the rug is laid down.

The most outstanding consequence of the change of method is a mathematical one. It now becomes possible to make lines at angles of 45° or 135° to the lines of the canvas with sharp definition. Thus diamonds can form part of a design. Octagons can also be made, and, further, very approximately regular octagons. If we take the usual convergents to the continued fraction for $\sqrt{2}$, viz., 3/2, 7/5, 17/12, 45/29, 99/70, etc., we proceed along the line of the canvas according to the numerator of one of these fractions, and obliquely according to its denominator. The choice of the fraction depends on the size of the rug, and in making a decision we have to note that the width or height of an octagon based on the convergent a/b is $a + 2b$, or the numerator of the next convergent. By removing from an octagon the isosceles right-angled triangles whose hypotenuses are the sides of the octagon we obtain an eight-pointed star. To obtain additional shapes we may divide and subdivide the foregoing ones.

The difficulty in making up a design from these elements is to dovetail them together while paying due regard to the main principle of a rug design. For a good rug should have a strongly marked central feature together with two side features of subsidiary interest.

The design shown in Fig. 1 is suitable for a hearth-rug of rough dimensions $54'' \times 27''$ made on canvas 96 stitches wide. The main outline consists of two regular octagons with their common side removed. The two large diamonds and the squares define the shape of the central star and the six large boat-shaped figures. The colour scheme is as follows. The black lines in the figure are meant to be reproduced in black wool in order to throw the various shapes into relief. The diamonds are dark brown, the small boats and stars are deep red, the six large boats and the central star are bright orange, while the squares are blue. There is a narrow border of black and the bounding rows have every stitch filled in. This rug has been made and is regarded as a satisfactory specimen of its kind.

It is of course impossible to make a rug with a smoothly curved

outline, but it is possible to give the illusion of smoothness. For example, to make the outline of a semicircle we need as many stitches as possible actually on the boundary or very nearly so. To do this we must find numbers which can be expressed as the sum of two
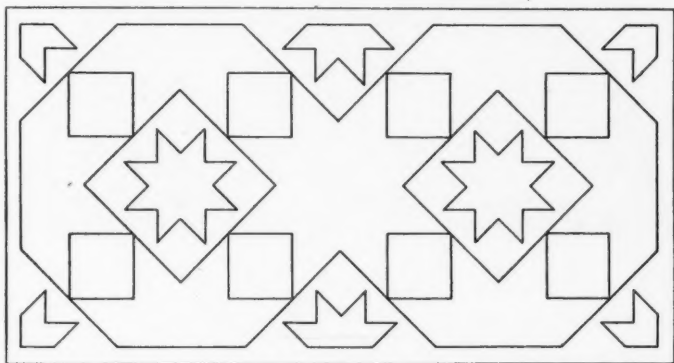
FIG. 1.

squares in more than one way. This is a systematic process based on the formula

$$(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (bc \mp ad)^2.$$

Thus
$$65 = 5 \times 13 = (1^2 + 2^2)(2^2 + 3^2) = 1^2 + 8^2 = 4^2 + 7^2.$$
$$325 = 5 \times 65 = (1^2 + 2^2)(1^2 + 8^2) = (1^2 + 2^2)(4^2 + 7^2)$$
$$= 1^2 + 18^2 = 6^2 + 17^2 = 10^2 + 15^2.$$
$$3250 = 1^2 + 57^2 = 15^2 + 55^2 = 21^2 + 53^2 = 35^2 + 45^2.$$

This last number is suitable for a small rug, about $36'' \times 18''$; for with the canvas usually supplied one yard = 15 big squares = 120 stitches.

A photograph of such a small rug is shown in Fig. 2 (p. 18). The design is too crowded—the star ought not to have satellites *—but the rug was made as an experiment and also to finish up some wool.

If a large hearth-rug is required, say about $63''$ long, the basic number is 11050.

$$\text{This} = 50 \times 13 \times 17 = 5^2 + 105^2 = 21^2 + 103^2 = 45^2 + 95^2$$
$$= 49^2 + 93^2 = 59^2 + 87^2 = 67^2 + 81^2.$$

If it is considered that a semicircular rug of this size will project too far from the fireplace, the design can be altered into an elliptic outline by multiplying the " ordinates " by some proper fraction,

* No doubt, since satellites belong to *planets*.

B
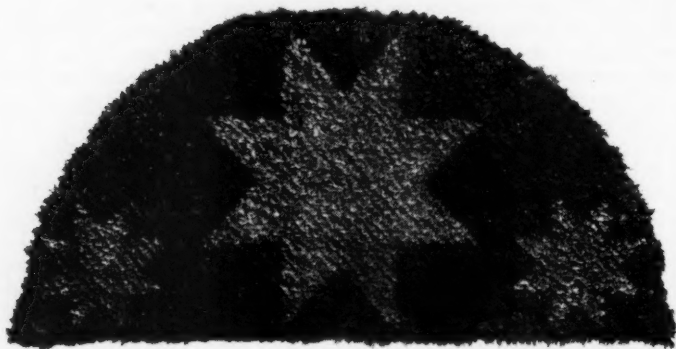
Photo: *Bone, Fakenham*.

Fig. 2.

say 4/5. We then get the following table of values for one quadrant :

x	-	105	103	95	93	87	81	67	59	49	45	21	5
y	-	4	17	36	39	47	54	65	70	74	76	82	84

The intervals between the leading stitches can then be " stepped up " as closely as possible.   N. M. G.

GLEANINGS FAR AND NEAR.

1337. When we speak of *Magnitude*, and say that any Quantity is great, we always make a Comparison. . . . Thus we say of an *Hill*, that 'tis Little ; or of a *Diamond*, that 'tis Large ; because we compare that Hill with others that are Higher, and in respect of them 'tis Little ; and we compare that Diamond with others that are Little, and in respect of them, we say 'tis a large one.— I. G. Pardies, *Short, but yet Plain Elements of Geometry*. Translated by J. Harris, 6th edition, 1725, p. 80 ; original edition in French, 1671. [Per Professor E. H. Neville.]

1338. *Large* is in fact a word which, standing by itself, has no more absolute meaning in mathematics than in the language of common sense. . . . 6 goals is a large score in a football match, but 6 runs is not a large score in a cricket match ; and 300 is a large score, but £300 is not a large income.—G. H. Hardy, *A Course of Pure Mathematics*, 1908, p. 113. [Per Professor E. H. Neville.]

1339. Also he does carpentry, because he has a bee in his bonnet about teaching geometry, and none of the kids are allowed to rest content with working their geometry on paper ; they have to go into the carpenter's shop and cut out all their triangles and things in wood. You should see the condition of the poor wretches who can see through propositions at a glance, but can't use a fret-saw to save their lives. They go nearly out of their minds.— G. D. H. and M. Cole, *Scandal at School*, p. 146 (1935). [Per Professor E. H. Neville.]

FORMULAE FOR NUMERICAL DIFFERENTIATION.

By W. G. BICKLEY.

In a recent paper (1) formulae were given for the numerical integration of a function in terms of its values at a set of arguments at equal intervals. In this companion paper, formulae for numerical differentiation, using the same data, are collected. Their utility in enabling derivatives of a function given numerically at such a set of arguments to be computed is obvious ; the need arises in several approximate methods which are coming more and more into use (2), (3). The formulae avoid the labour of preliminary differencing, and are indeed more convenient than the finite difference formulae when the derivative is required at all the points of subdivision of a limited range.

Notation.

If $y = f(x)$ is a function of x, the value of y at $x_p = x_0 + ph$ will be denoted by y_p ; h is the tabular interval, and p will (almost everywhere) denote an integer. With D denoting the differential operator, $D^m y_p$ will denote the mth derivative of y at x_p.

Basic formula and description of Tables (pp. 22-27).

The Tables give, for a set of values of the various parameters, the numerical values of $_{mn}A_{pr}$ and $_{mn}E_p$ in the formula

$$\frac{h^m D^m y_p}{m!} = \frac{1}{n!} \sum_{r=0}^{n} {}_{mn}A_{pr} y_r + {}_{mn}E_p.$$

(In the Tables, the suffixes m, n, and p are dropped, A_r and E simply being sufficient, since m, n, and p are explicitly given.) E is the " error " term, and is normally a multiple of $h^{n+1} D^{n+1} y(X)$, where X is some (unspecified) value of x between x_0 and x_n. Occasionally E is of a higher order ; such cases are clearly indicated.

For $n = 2(1)6$ the complete set of coefficients (*i.e.*, $p = 0(1)n = m$) is given ; for $n = 8$ and 10 advantage is taken of the symmetry of these matrices to save space. Also, in the latter cases, coefficients for the first four derivatives only are given ; if higher derivatives are required they may be obtained by repeated applications of the formula. For instance, $D^6 y$ can be computed by first finding $D^4 y_p = z_p$ for $p = 0(1)n$, and then computing $D^2 z_p$.

Calculation and checking of coefficients.

It is clear that the A are multiples of the derivatives of the Lagrangian interpolation polynomials for the values x_p of x. For $n = 2(1)6$ the coefficients were in fact calculated from these polynomials. For $n = 7(1)10$ use was made of the formula

$$_{mn}A_{pr} = n_{m,\,n-1}A_{pr} + (-)^{n-r}\binom{n}{r}{}_{mn}A_{pn}.$$

The value of $_{mn}A_{pn}$ were obtained from tables prepared in connection with another investigation.

This formula was also used to check the coefficients for $n = 2(1)6$. For $n = 8$ and 10 (in calculating which the values for 7 and 9 had also to be obtained) the symmetry (or anti-symmetry) of the complete matrices was an almost conclusive check—after the few errors which it disclosed had been corrected, no further error was detected.

Finally, the proofs have been checked by means of the equations

$$\sum_{r=0}^{n} {}_{mn}A_{pr} = 0, \quad \sum_{r=0}^{n} r_{mn}A_{pr} = \begin{array}{l} n! \ (m=1) \\ 0 \ \ (m>1) \end{array},$$

which are the results of applying the formulae to the functions $y = 1$ and $y = x$ with $h = 1$.

The author wishes to thank Dr. J. C. P. Miller for an additional check.

A further application of the formulae.

The author's original idea as to the use of these formulae was their application to the step-by-step numerical integration of differential equations. Choosing n and h so that the error term is sufficiently small to justify the expectation that the desired degree of accuracy may be achieved, suppose that $y_0, y_1, \ldots y_{n-1}$ have been determined (e.g., by Maclaurin's series). The successive derivatives of y at x_n are expressed in terms of $y_0, y_1, \ldots y_n$, and these expressions are substituted in the differential equation, which thereby becomes an algebraic equation for y_n.

Having thus determined y_n, we use $y_1, y_2, \ldots y_n$ to determine $y_{n+1} \ldots$, and so on.

The method is susceptible of elaboration and refinement, but we content ourselves here with a crude application to a very simple differential equation, namely to approximate to e^x as a solution of the equation $Dy = y$, taking $h = 0 \cdot 1$.

By 3–1–3,
$$0 \cdot 1 Dy_3 = (11y_3 - 18y_2 + 9y_1 - 2y_0)/6$$

or, since $Dy_3 = y_3$,
$$10 \cdot 4 y_3 = 18y_2 - 9y_1 + 2y_0.$$

Similarly, $\qquad\qquad 10 \cdot 4 y_4 = 18y_3 - 9y_2 + 2y_1,$
and so on.

If we apply the more accurate formula 4–1–4, we find
$$47 \cdot 6 y_4 = 96y_3 - 72y_2 + 32y_1 - 6y_0.$$

A start was made by use of the eight decimal values of e^x obtained from tables; they are given in column 1. Columns 2 and 3

give the results of successive applications of 3–1–3 and 4–1–4 respectively.

x	1 e^x	2 3–1–3	3 4–1–4
0·0	1·00000 000		
0·1	1·10517 092		
0·2	1·22140 276		
0·3	1·34985 881	1·34987 610	
0·4	1·49182 470	1·49187 373	1·49182 598
0·5	1·64872 127	1·64881 228	1·64872 527
0·6	1·82211 880	1·82226 054	1·82212 648
0·7	2·01375 271	2·01395 449	2·01376 473
0·8	2·22554 093	2·22581 352	2·22555 800
0·9	2·45960 311	2·45995 904	2·45962 612
1·0	2·71828 183	2·71873 558	2·71831 185

Enough decimals have been carried to ensure that the error is not obscured by " rounding-off errors ". With 4–1–4, the error E at any stage is less than $10^{-5}y_n/5$—since $D^5y = y$—so that the sum of the errors due to the steps from y_4 to y_{10} is less than

$$\frac{1}{5} \times \frac{24}{47\cdot6} \times 10^{-5} \overset{10}{\underset{4}{\Sigma}} y_n = 1\cdot44 \times 10^{-5}.$$

The actual error at $x = 1\cdot0$ turns out to be about double this,

$$(3\cdot0 \ldots \times 10^{-5}).$$

The discrepancy is due, of course, to the cumulative effect of the errors of the approximation made in previous lines. It is possible to obtain a check which will indicate roughly the amount of the error at each stage, and so to correct it, and the elaborated and refined —but more laborious—procedure can be expected to give considerably reduced errors.

REFERENCES.

(1) W. G. Bickley, " Formulae for Numerical Integration ". *Math. Gazette*, XXIII, pp. 352-359, Oct. 1939.

(2) K. N. E. Bradfield and R. V. Southwell, " Relaxation Methods Applied to Engineering Problems ". *Proc. Roy. Soc.*, A, 161, pp. 155-181, July 1937.

(3) P. D. Crout, " An Application of Polynomial Approximation to the Solution of Integral Equations arising in Physical Problems ". *J. Math. and Phys.*, XIX, pp. 34-91, Jan. 1940. (Crout gives coefficients for $m = 1$, 2, and 3, with $n = 2$, 4, and 6.)

n	m	p	A_0	A_1	A_2	A_3	E
		0	-3	4	-1		$+1/3 \;.\; h^3 f^{iii}$
2	1	1	-1	0	1		$-1/6$
		2	1	-4	3		$+1/3$
		0	1	-2	1		$-1/2$
2	2	1	1	-2	1		$-1/24 \;.\; h^4 f^{iv}$
		2	1	-2	1		$+1/2 \;.\; h^3 f^{iii}$
		0	-11	18	-9	2	$-1/4 \;.\; h^4 f^{iv}$
3	1	1	-2	-3	6	-1	$+1/12$
		2	1	-6	3	2	$-1/12$
		3	-2	9	-18	11	$+1/4$
		0	6	-15	12	-3	$+11/24$
3	2	1	3	-6	3	0	$-1/24$
		2	0	3	-6	3	$-1/24$
		3	-3	12	-15	6	$+11/24$
		0	-1	3	-3	1	$-1/4$
3	3	1	-1	3	-3	1	$-1/12$
		2	-1	3	-3	1	$+1/12$
		3	-1	3	-3	1	$+1/4$

n	m	p	A_0	A_1	A_2	A_3	A_4	E
		0	-50	96	-72	32	-6	$+1/5 \;.\; h^5 f^{v}$
		1	-6	-20	36	-12	2	$-1/20$
4	1	2	2	-16	0	16	-2	$+1/30$
		3	-2	12	-36	20	6	$-1/20$
		4	6	-32	72	-96	50	$+1/5$
		0	35	-104	114	-56	11	$-5/12$
		1	11	-20	6	4	-1	$+1/24$
4	2	2	-1	16	-30	16	-1	$+1/180 \;.\; h^6 f^{vi}$
		3	-1	4	6	-20	11	$-1/24 \;.\; h^5 f^{v}$
		4	11	-56	114	-104	35	$+5/12$
		0	-10	36	-48	28	-6	$+7/24$
		1	-6	20	-24	12	-2	$+1/24$
4	3	2	-2	4	0	-4	2	$-1/24$
		3	2	-12	24	-20	6	$+1/24$
		4	6	-28	48	-36	10	$+7/24$
		0	1	-4	6	-4	1	$-1/12$
		1	1	-4	6	-4	1	$-1/24$
4	4	2	1	-4	6	-4	1	$-1/144 \;.\; h^6 f^{vi}$
		3	1	-4	6	-4	1	$+1/24 \;.\; h^5 f^{v}$
		4	1	-4	6	-4	1	$+1/12$

n	m	p	A_0	A_1	A_2	A_3	A_4	A_6	E
5	1	0	−274	600	−600	400	−150	24	$-1/6 \cdot h^6 f^{vi}$
		1	−24	−130	240	−120	40	−6	$+1/30$
		2	6	−60	−40	120	−30	4	$-1/60$
		3	−4	30	−120	40	60	−6	$+1/60$
		4	6	−40	120	−240	130	24	$-1/30$
		5	−24	150	−400	600	−600	274	$+1/6$
5	2	0	225	−770	1070	−780	305	−50	$+137/360$
		1	50	−75	−20	70	−30	5	$-13/360$
		2	−5	80	−150	80	−5	0	$+1/180$
		3	0	−5	80	−150	80	−5	$+1/180$
		4	5	−30	70	−20	−75	50	$-13/360$
		5	−50	305	−780	1070	−770	225	$+137/360$
5	3	0	−85	355	−590	490	−205	35	$-5/16$
		1	−35	125	−170	110	−35	5	$-1/48$
		2	−5	−5	50	−70	35	−5	$+1/48$
		3	5	−35	70	−50	5	5	$-1/48$
		4	−5	35	−110	170	−125	35	$+1/48$
		5	−35	205	−490	590	−355	85	$+5/16$
5	4	0	15	−70	130	−120	55	−10	$+17/144$
		1	10	−45	80	−70	30	−5	$+5/144$
		2	5	−20	30	−20	5	0	$-1/144$
		3	0	5	−20	30	−20	5	$-1/144$
		4	−5	30	−70	80	−45	10	$+5/144$
		5	−10	55	−120	130	−70	15	$+17/144$
5	5	0	−1	5	−10	10	−5	1	$-1/48$
		1	−1	5	−10	10	−5	1	$-1/80$
		2	−1	5	−10	10	−5	1	$-1/240$
		3	−1	5	−10	10	−5	1	$+1/240$
		4	−1	5	−10	10	−5	1	$+1/80$
		5	−1	5	−10	10	−5	1	$+1/48$

n	m	p	A_0	A_1	A_2	A_3	A_4	A_5	A_6	E
		0	−1764	4320	−5400	4800	−2700	864	−120	$+1/7$. $h^7 f$ vii
		1	−120	−924	1800	−1200	600	−180	24	$−1/42$
		2	24	−288	−420	960	−360	96	−12	$+1/105$
6	1	3	−12	108	−540	0	540	−108	12	$−1/140$
		4	12	−96	360	−960	420	288	−24	$+1/105$
		5	−24	180	−600	1200	−1800	924	120	$−1/42$
		6	120	−864	2700	−4800	5400	−4320	1764	$+1/7$
		0	1624	−6264	10530	−10160	5940	−1944	274	$−7/20$
		1	274	−294	−510	940	−570	186	−26	$+11/360$
		2	−26	456	−840	400	30	−24	4	$−1/180$
6	2	3	4	−54	540	−980	540	−54	4	$−1/1120$. $h^8 f$ viii
		4	4	−24	30	400	−840	456	−26	$+1/180$. $h^7 f$ vii
		5	−26	186	−570	940	−510	−294	274	$−11/360$
		6	274	−1944	5940	−10160	10530	−6264	1624	$+7/20$
		0	−735	3480	−6915	7440	−4605	1560	−225	$+29/90$
		1	−225	840	−1245	960	−435	120	−15	$+7/720$
		2	−15	−120	525	−720	435	−120	15	$−1/90$
6	3	3	15	−120	195	0	−195	120	−15	$+7/720$
		4	−15	120	−435	720	−525	120	15	$−1/90$
		5	15	−120	435	−960	1245	−840	225	$+7/720$
		6	225	−1560	4605	−7440	6915	−3480	735	$+29/90$
		0	175	−930	2055	−2420	1605	−570	85	$−7/48$
		1	85	−420	855	−920	555	−180	25	$−1/36$
		2	25	−90	105	−20	−45	30	−5	$+1/144$
6	4	3	−5	60	−195	280	−195	60	−5	$+7/5760$. $h^8 f$ viii
		4	−5	30	−45	−20	105	−90	25	$−1/144$. $h^7 f$ vii
		5	25	−180	555	−920	855	−420	85	$+1/36$
		6	85	−570	1605	−2420	2055	−930	175	$+7/48$
		0	−21	120	−285	360	−255	96	−15	$+5/144$
		1	−15	84	−195	240	−165	60	−9	$+1/72$
		2	−9	48	−105	120	−75	24	−3	$+1/720$
6	5	3	−3	12	−15	0	15	−12	3	$−1/360$
		4	3	−24	75	−120	105	−48	9	$+1/720$
		5	9	−60	165	−240	195	−84	15	$+1/72$
		6	15	−96	255	−360	285	−120	21	$+5/144$
		0	1	−6	15	−20	15	−6	1	$−1/240$
		1	1	−6	15	−20	15	−6	1	$−1/360$
		2	1	−6	15	−20	15	−6	1	$−1/720$
6	6	3	1	−6	15	−20	15	−6	1	$−1/2880$. $h^8 f$ viii
		4	1	−6	15	−20	15	−6	1	$+1/720$. $h^7 f$ vii
		5	1	−6	15	−20	15	−6	1	$+1/360$
		6	1	−6	15	−20	15	−6	1	$+1/240$

n	m	p	A_0	A_1	A_2	A_3	A_4		E
		0	$-1\,09584$	$3\,22560$	$-5\,64480$	$7\,52640$	$-7\,05600$	8	$+1/9 \; . \; h^9 f^{ix}$
		1	-5040	-64224	$1\,41120$	$-1\,41120$	$1\,17600$	7	$-1/72$
		2	720	-11520	-38304	80640	-50400	6	$+1/252$
		3	-240	2880	-20160	-18144	50400	5	$-1/504$
8	1	4	144	-1536	8064	-32256	0	4	$+1/630$
		5	-144	1440	-6720	20160	-50400	3	$-1/504$
		6	240	-2304	10080	-26880	50400	2	$+1/252$
		7	-720	6720	-28224	70560	$-1\,17600$	1	$-1/72$
		8	5040	-46080	$1\,88160$	$-4\,51584$	$7\,05600$	0	$+1/9$
		0	-67284	$3\,90880$	$-10\,27768$	$16\,06752$	$-16\,31840$	8	$+29531/90720$
		1	-13132	50904	-81872	75320	-47880	7	$-1/5670$
		2	-140	-11872	45864	-70112	57680	6	$-331/90720$
		3	252	-2408	-2800	24696	-38360	5	$+59/22680$
8	3	4	-196	2016	-9464	13664	0	4	$-41/18144$
		5	196	-1960	9072	-25928	38360	3	$+59/22680$
		6	-252	2464	-11032	30240	-57680	2	$-331/90720$
		7	140	-1512	7504	-22792	47880	1	$-1/5670$
		8	13132	$-1\,18048$	$4\,71240$	$-10\,95584$	$16\,31840$	0	$+29531/90720$
			$-A_8$	$-A_7$	$-A_6$	$-A_5$	$-A_4$	p	

n	m	p	A_0	A_1	A_2	A_3	A_4		E
		0	$1\,18124$	$-5\,54112$	$12\,51936$	$-17\,94688$	$17\,41320$	8	$-761/2520 \; . \; h^9 f^{ix}$
		1	13068	512	-83664	$1\,54224$	$-1\,48120$	7	$+223/10080$
		2	-1044	22464	-37072	$.\,4032$	22680	6	$-19/5040$
		3	188	-2736	29232	-52864	27720	5	$+1/1120$
8	2	4	-36	512	-4032	32256	-57400	4	$+1/6300 \; . \; h^{10} f^{x}$
		5	-36	288	-784	-1008	27720	3	$-1/1120 \; . \; h^9 f^{ix}$
		6	188	-1728	7056	-16576	22680	2	$+19/5040$
		7	-1044	9584	-39312	94752	$-1\,48120$	1	$-223/10080$
		8	13068	$-1\,18656$	$4\,80032$	$-11\,37024$	$17\,41320$	0	$+761/2520$
		0	22449	$-1\,47392$	$4\,28092$	$-7\,20384$	$7\,69510$	8	$-89/480$
		1	6769	-38472	96292	$-1\,40504$	$1\,32510$	7	$-101/5760$
		2	889	-1232	-6468	21616	-28490	6	$+13/2880$
		3	-231	2968	-9548	12936	-7490	5	$-7/5760$
8	4	4	49	-672	4732	-13664	19110	4	$-41/181440 . h^{10} f^{x}$
		5	49	-392	1092	616	-7490	3	$+7/5760 \; . \; h^9 f^{ix}$
		6	-231	2128	-8708	20496	-28490	2	$-13/2880$
		7	889	-8232	34132	-83384	$1\,32510$	1	$+101/5760$
		8	6769	-60032	$2\,35452$	$-5\,34464$	$7\,69510$	0	$+89/480$
			A_8	A_7	A_6	A_5	A_4	p	

n	m	p	A_0	A_1	A_2	A_3	A_4	A_5	p	E
10	1	0	-106 28640	362 88000	-816 48000	1451 52000	-1905 12000	1828 91520	10	$+\frac{1}{11}.h^{11}\int x1$
		1	-3 62880	-66 36960	163 29600	-217 72800	254 01600	-228 61440	9	$-1/110$
		2	40320	-8 06400	-44 19360	96 76800	-84 67200	67 73760	8	$+1/495$
		3	-10080	1 51200	-13 60800	-27 56160	63 50400	-38 10240	7	$-1/1320$
		4	4320	-57600	3 88800	-20 73600	-13 30560	43 54560	6	$+1/2310$
		5	-2880	36000	-2 16000	8 64000	-30 24000	0	5	$-1/2772$
		6	2880	-34560	1 94400	-6 91200	18 14400	-43 54560	4	$+1/2310$
		7	-4320	50400	-2 72160	9 07200	-21 16800	38 10240	3	$-1/1320$
		8	10080	-1 15200	6 04800	-19 35360	42 33000	-67 73760	2	$+1/495$
		9	-40320	4 53600	-23 32800	72 57600	-152 40960	228 61440	1	$-1/110$
		10	3 62880	-40 32000	204 12000	-622 08000	1270 08000	-1828 91520	0	$+1/11$
10	3	0	-84 09500	575 37360	-1877 95260	3845 55840	-5419 68840	5429 59200	10	$+16103/50400$
		1	-11 72700	44 90200	-69 61140	57 00240	-24 35160	-1 81440	9	$-41/11200$
		2	8540	-126 66640	49 59900	-83 70240	85 18440	-63 80640	8	$-593/453600$
		3	7900	-78360	-8 32140	36 56400	-57 63240	48 68640	7	$+263/302400$
		4	-5340	66640	-3 72060	4 89600	18 94200	-32 96160	6	$-13/21600$
		5	4100	-50440	2 92140	-10 48560	14 01960	0	5	$+479/907200$
		6	-4100	49200	-2 75940	9 68640	-24 01560	32 96160	4	$-13/21600$
		7	5340	-62840	3 42900	-11 57040	27 30840	-48 68640	3	$+263/302400$
		8	-7900	92240	-4 97340	16 46400	-37 64040	63 80640	2	$-593/453600$
		9	-8540	86040	-3 77460	9 11760	-111 71800	1 81440	1	$-41/11200$
		10	11 72700	-129 08240	6454 84540	-1938 72960	3879 02760	-5429 59200	0	$+16103/50400$
			$-A_{10}$	$-A_9$	$-A_8$	$-A_7$	$-A_6$	$-A_5$		

n	m	p	A_0	A_1	A_2	A_3	A_4	A_5	p	E
		0	127 53576	−699 98400	1983 20400	−3767 61600	5103 75600	−4991 05152	10	$-671/2520 \cdot h^{11} f^{xi}$
		1	10 26576	146 61240	−135 36720	289 35360	−379 91520	360 97488	9	$+419/25200$
		2	−69264	17 88480	−23 48280	−21 08160	60 78240	−59 91552	8	$-31/12600$
		3	11016	−1 90440	23 94360	−41 65920	15 27120	9 88848	7	$+29/50400$
		4	−2664	40320	−3 39960	28 33920	−50 45040	27 57888	6	$-1/6300$
10	2	5	576	−9000	72000	−4 32000	30 24000	−53 11152	5	$-1/33264 \cdot h^{12} f^{xii} / h^{11} f^{xi}$
		6	576	−5760	22680	−23040	−2 41920	27 57888	4	$+1/6300 \cdot h^{11} f^{xi}$
		7	−2664	29880	−1 52280	4 62240	−9 02160	9 88848	3	$-29/50400$
		8	11016	−1 23840	6 35760	−19 69920	40 97520	−59 91552	2	$+31/12600$
		9	−69264	7 72920	−39 33360	120 64320	−248 27040	360 97488	1	$-419/25200$
		10	10 26576	−113 61600	572 34600	−1733 18400	3508 34400	−4991 05152	0	$+671/2520$
		0	34 16930	−265 57640	953 16120	−2081 94720	3060 06540	−3152 46960	10	$-7645/36288 \cdot h^{11} f^{xi}$
		1	7 23080	−45 43550	132 44760	−240 91080	306 19680	−283 33620	9	$-2041/181440 \cdot h^{12} f^{xii} / h^{11} f^{xi}$
		2	50840	1 64440	−17 47350	48 56160	−73 13880	71 31600	8	$+167/60480$
		3	−1 2790	1 91530	−5 39010	3 63000	6 35460	−14 04900	7	$-277/362880$
		4	3590	−5 2280	3 88980	−11 31360	1 54 7700	−10 23120	6	$+41/181440$
10	4	5	−820	1 2610	−97380	5 24280	−14 01960	19 26540	5	$+479/10886400 \cdot h^{12} f^{xii} / h^{11} f^{xi}$
		6	−820	8200	−32490	37920	2 53680	−10 23120	4	$-41/181440 \cdot h^{11} f^{xi}$
		7	3590	−40310	2 05650	−6 24840	1 2 22620	−14 04900	3	$+277/362880$
		8	−1 2790	1 44280	−7 43760	23 16000	−48 45540	71 31600	2	$-167/60480$
		9	50840	−5 72030	29 49480	−91 32360	190 93200	−283 33620	1	$+2041/181440$
		10	7 23680	−79 09640	392 30370	−1164 66720	2296 82040	−3152 46960	0	$+7645/36288$
n	m	p	A_{10}	A_9	A_8	A_7	A_4	A_5	p	

PONCELET'S PORISTIC POLYGONS.

By F. H. V. Gulasekharam.

Professor T. W. Chaundy has published two papers on the above subject in Vols. 22 and 25 of the *Proceedings of the London Mathematical Society*. He discusses the condition necessary that a single infinitude of n-sided polygons may be inscribed in a circle S and circumscribed to another circle S', basing his discussion on the theory of elliptic functions. A treatment of the topic, *within the scope of elementary mathematics*, is indicated in this note.

2. Suppose S is a circle with centre O and radius R. Let $\sigma_1, \sigma_2, \sigma_3$ be three circles of radii ρ_1, ρ_2, ρ_3 respectively, coaxal with S, and having their centres at distances $\delta_1, \delta_2, \delta_3$ from O. Then if $\frac{1}{2}h$ be the distance of O from the radical axis of the system, it is well known that

$$R^2 + \delta_k{}^2 - \rho_k{}^2 = h\delta_k. \qquad\qquad\qquad (2\cdot1)$$

A Lemma:

We shall now prove that if a triangle ABC is inscribed in S such that the sides BC, CA, AB touch $\sigma_1, \sigma_2, \sigma_3$ respectively, then

$$[R^2 \Sigma \delta_1 - \delta_1 \delta_2 \delta_3]^2 = 4R^2 [R^2 \Sigma \delta_2 \delta_3 - h\delta_1 \delta_2 \delta_3]. \qquad\qquad (2\cdot2)$$

Proof: Let p, q, r be the perpendicular distances of A, B, C from the radical axis. Then it is well known that the powers of A, B, C with respect to the circle σ_k are

$$2p\delta_k, \quad 2q\delta_k, \quad 2r\delta_k$$

respectively.

Hence, $\sqrt{(2q\delta_1)} + \sqrt{(2r\delta_1)} = a,$

or $\sqrt{2}(\sqrt{q} + \sqrt{r}) = a/\sqrt{\delta_1},$

Similarly, $\sqrt{2}(\sqrt{r} + \sqrt{p}) = b/\sqrt{\delta_2},$

and $\sqrt{2}(\sqrt{p} + \sqrt{q}) = c/\sqrt{\delta_3},$

where a, b, c denote as usual the lengths of the sides of $\triangle ABC$.

Hence

$$\left.\begin{array}{l} 2\sqrt{2}\sqrt{p} = -a/\sqrt{\delta_1} + b/\sqrt{\delta_2} + c/\sqrt{\delta_3}, \\ 2\sqrt{2}\sqrt{q} = a/\sqrt{\delta_1} - b/\sqrt{\delta_2} + c/\sqrt{\delta_3}, \\ 2\sqrt{2}\sqrt{r} = a/\sqrt{\delta_1} + b/\sqrt{\delta_2} - c/\sqrt{\delta_3}. \end{array}\right\} \qquad\qquad (2\cdot3)$$

Now since $\frac{1}{2}h$ is the distance of the radical axis from O, which is the centre of mean position for multiples $(1 - \cot B \cot C)$ at A, $(1 - \cot C \cot A)$ at B, and $(1 - \cot A \cot B)$ at C, we have

$$h = \Sigma p (1 - \cot B \cot C).$$

Hence substituting for p, q, r from (2·3),

$$4h = \Sigma a^2/\delta_1 - 2\Sigma bc \cot B \cot C/\sqrt{(\delta_2 \delta_3)},$$

or $h/R^2 = \Sigma (1 - \cos^2 A)/\delta_1 - 2\Sigma \cos B \cos C/\sqrt{(\delta_2 \delta_3)}.$

Hence

$$1/\delta_1 + 1/\delta_2 + 1/\delta_3 - h/R^2 = (\cos A/\sqrt{\delta_1} + \cos B/\sqrt{\delta_2} + \cos C/\sqrt{\delta_3})^2. \quad \ldots(2\cdot4)$$

Again we have the identity

$$2M = \Sigma(q - r)^2 \cot A,$$

where M is the area of $\triangle ABC$.

Substituting for p, q, r from (2·3) and writing

$$M = 2R^2 \sin A \sin B \sin C,$$

we have

$$16R^2 \sin A \sin B \sin C = \Sigma \frac{a^2}{\delta_1} \left[\frac{b^2}{\delta_2} + \frac{c^2}{\delta_3} - \frac{2bc}{\sqrt{(\delta_2 \delta_3)}} \right] \cot A.$$

Now writing $a = 2R \sin A$, etc., and noting that

$$\Sigma(\sin A \cos A/\delta_1)(\sin^2 B/\delta_2 + \sin^2 C/\delta_3)$$
$$= \Sigma(1/\delta_2\delta_3) \sin A \sin B \sin C,$$

we get

$$[R^2 \Sigma(1/\delta_2\delta_3) - 1] \sqrt{(\delta_1\delta_2\delta_3)}/2R^2$$
$$= \Sigma(\cos A/\sqrt{\delta_1}). \quad \ldots\ldots\ldots\ldots\ldots\ldots(2\cdot5)$$

Hence, equating the two expressions for $\Sigma[\cos A/\sqrt{\delta_1}]$ in (2·4) and (2·5), we have the relation (2·2).

3. Let us now consider a polygon of n sides, $A_1A_2 \ldots A_n$, inscribed in a circle S, centre O and radius R, and circumscribed to the circle S' of radius r, whose centre is at a distance d from O.

If we consider the polygon of $(n + 1)$ sides, $A_1A_2A_3 \ldots A_nA_1$, whose $(n + 1)$th side is the limiting position of the chord $A_{n+1}A_1$ of S when $A_{n+1} \to A_1$, we see by repeated application of Poncelet's theorem that the chords

$$A_1A_2, \ A_1A_3, \ A_1A_4, \ \ldots, \ A_1A_n, \ A_1A_1,$$

respectively touch the *coaxal* circles

$$S_1 \equiv S', \ S_2, \ S_3, \ \ldots, \ S_{n-1} \equiv S', \ S_n \equiv S,$$

whose radii are

$$r_1 = r, \ r_2, \ r_3, \ \ldots, \ r_{n-1} = r, \ r_n = R,$$

and the distances of whose centres from O are

$$d_1 = d, \ d_2, \ d_3, \ \ldots, \ d_{n-1} = d, \ d_n = 0.$$

The same is true of the chords joining any vertex A_k to the successive vertices

$$A_{k+1}, \ A_{k+2}, \ \ldots, \ A_n, \ A_1, \ A_2, \ \ldots, \ A_k$$

in that order.

Writing $d_k = RD_k^2$, we shall in the next section establish certain *Recurrence Formulae*, which will enable us to express d_2, d_3, $\ldots d_n$ in terms of R, d, and r. *The condition for porism is $d_n = 0$, that is,* $D_n = 0$.

4. *Recurrence Formulae for D_{2m+1} and D_{2m}.*

Let (i, j) denote the chord joining the vertices $A_i A_j$ of the polygon. Then the chords $(i, i+k)$, $(i, i+m)$, $(i, i+m+k)$, $(i+k, i+m+k)$, $(i+k, i+m)$, where $m > k$, touch the circles S_k, S_m, S_{m+k}, S_m, S_{m-k} of § 3.

Hence considering the triangles

$$A_i A_{i+k} A_{i+m+k} \quad \text{and} \quad A_i A_{i+k} A_{i+m},$$

we see that if a triangle is inscribed in S such that two of its sides touch S_m and S_k, then the third side touches either S_{m+k} or S_{m-k}. If, therefore, we put $\delta_1 = R D_m^2$, $\delta_2 = R D_k^2$, and $\delta_3 = R T^2$ in equation (2·2), we see that D_{m+k}^2 and D_{m-k}^2 are the two values of T^2 given by the resulting quadratic equation, which with the aid of (2·1) can be written in the form

$$(1 - D_m^2 D_k^2)^2 T^4 - 2(D_k^2 r_m^2 + D_m^2 r_k^2)(T/R)^2$$
$$+ (D_m^2 - D_k^2)^2 = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4·1)$$

Hence, when $m > k$,

$$D_{m+k} D_{m-k} = (D_m^2 - D_k^2)/(1 - D_m^2 D_k^2). \quad \ldots\ldots\ldots(4·2)\,*$$

When $m = k$, one of the values of T^2 is zero, and then from (4·1),

$$D_{2m} = 2 D_m (r_m/R)/(1 - D_m^4). \quad \ldots\ldots\ldots\ldots\ldots(4·3)\,*$$

Deductions from (4·2) :

$$D_{m+1} D_{m-1} = (D_m^2 - D_1^2)/(1 - D_m^2 D_1^2), \quad \ldots\ldots\ldots\ldots(4·4)$$

$$D_1 D_{2m+1} = (D_{m+1}^2 - D_m^2)/(1 - D_m^2 D_{m+1}^2), \quad \ldots\ldots\ldots(4·5)$$

$$D_2 D_{2m} = (D_{m+1}^2 - D_{m+1}^2)/(1 - D_{m+1}^2 D_{m-1}^2). \quad \ldots\ldots\ldots(4·6)$$

When D_1 and D_2 are known, D_3, D_4, D_5, etc., can be calculated in succession from (4·4) ; obviously much labour can be saved by using (4·5) and (4·6) in place of (4·4).

5. *Expressions for D_2, D_3, D_4, etc. ; r_2, r_3, r_4.*

We have $\qquad\qquad D_1 = \sqrt{(d/R)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5·1)$

From (4·3), $\qquad\quad D_2 = 2r\sqrt{(Rd)}/(R^2 - d^2). \quad \ldots\ldots\ldots\ldots(5·2)$

Then from (4·5) and (4·6), we get

$$D_3 = - D_1 M_3 M_3'/(N_3 N_3'), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5·3)$$

$$D_4 = 4r\sqrt{(Rd)}(R^2 - d^2) M_4/(N_4 N_4'), \quad \ldots\ldots\ldots\ldots\ldots\ldots(5·4)$$

$$D_5 = D_1 M_5 M_5'/(N_5 N_5'), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5·5)$$

$$D_6 = D_2 M_3 M_3' N_3 N_3' M_6/(N_6 N_6'), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5·6)$$

$$D_7 = - D_1 M_7 M_7'/(N_7 N_7'), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5·7)$$

$$D_8 = 8r\sqrt{(Rd)}(R^2 - d^2) M_4 N_4 N_4' M_8/(N_8 N_8') ; \quad \ldots\ldots\ldots\ldots(5·8)$$

---

* The ambiguity of signs on the left-hand side of equations (4·2) and (4·3) may be supposed absorbed in D_{m+k}, D_{m-k}, D_{2k} as the case may be.

where $M_3 = R^2 - d^2 - 2Rr$,

$M_4 = (R^2 - d^2)^2 - 2r^2(R^2 + d^2)$,

$N_4 = (R^2 - d^2)^2 - 4r^2Rd$,

$M_5 = (R^2 - d^2) M_3 M_3' + 2RrN_3N_3'$,

$M_6 = 3(R^2 - d^2)^4 - 4r^2(R^2 + d^2)(R^2 - d^2)^2 - 16r^4R^2d^2$,

$N_6 = N_4'^2 - 16r^4Rd(R + d)^2$,

$M_7 = 4rR(R^2 - d^2) N_3N_3'M_4 - M_3M_3'N_4N_4'$,

$M_8 = N_4^2N_4'^2 - 2M_4^2[(R^2 - d^2)^4 + 16r^4R^2d^2]$,

$N_8 = N_4^2N_4'^2 - 16r^2Rd(R^2 - d^2)^2M_4^2$.

M_3' M_5', M_7', N_4', N_6', N_8' are obtained from M_3, M_5, M_7, N_4, N_6, N_8 respectively by changing the sign of R; while N_3, N_3', N_5, N_5', N_7, N_7' are obtained from M_3, M_3', M_5, M_5', M_7, M_7' respectively by interchanging R and d.

Again from (5·4) and (4·3),

$$r_2/R = M_4/(R^2 - d^2)^2 \quad \text{...........................(5·9)}$$

$$= 1 - [r/(R + d)]^2 - [r/(R - d)]^2. \quad \text{.........(5·10)}$$

From (5·6) and (4·3) we get

$$r_3/R = M_6/(N_3N_3')^2. \quad \text{............................(5·11)}$$

Similarly,

$$r_4/R = M_8/(N_4N_4')^2. \quad \text{............................(5·12)}$$

We should remark that we can get D_{2k} by replacing d by d_2 and r by r_2 in the expression for D_k. Similarly for r_{2k}, D_{3k}, etc.

For example, we can get (5·12) from (5·9), noting that

$$R^2 - d_2^2 = R^2N_4N_4'/(R^2 - d^2)^4, \quad \text{.................(5·13)}$$

and $\quad R^2 + d_2^2 = R^2[(R^2 - d^2)^4 + 16r^4R^2d^2]/(R^2 - d^2)^4. \quad \text{.......(5·14)}$

6. Poristic conditions.

The condition for porism of n sides is $D_n = 0$. But care must be taken that no extraneous condition which makes $D_j = 0$ $(j < n)$ is included in the condition $D_n = 0$.

The condition $D_{2m+1} = 0$ *is equivalent to* $D_{m+1} = \pm D_m$; *while* $D_{2m} = 0$ *is equivalent to* $D_{m+1} = \pm D_{m-1}$.

If, however, we use (4·3), *the condition* $D_{2m} = 0$ *is equivalent to*

either (i) $r_m = 0$,

or (ii) $D_m \to \infty$.

These conditions prove the following geometrical property:

If a polygon of an even number of sides is inscribed in a circle S and circumscribed to another circle S', then the diagonals joining each pair of opposite vertices EITHER *intersect at a limiting point of the coaxal system of circles of which S and S' are members,* OR *are parallel to the radical axis of S and S'.*

Let us denote by $P'[n]$ the poristic condition corresponding to $D_{\frac{1}{2}n} \to \infty$, when n is even. In all other cases, let $P[n]$ denote the condition for porism of an n-sided polygon.

$P\,[2m+1]$ will be found to be of the form $M_{2m+1}M'_{2m+1}=0$, while $P'\,[2m]$ is of the form $N_{2m}N'_{2m}=0$. (*Vide* § 5.) To save space, we shall in the succeeding sections write $M_{2m+1}=0$ for $P\,[2m+1]$, and $N_{2m}=0$ for $P'\,[2m]$, it being understood that in the former case an alternative condition is obtained *by changing the sign of R*, and in the latter case *by changing the sign of d*.

7. $P\,[n]$.

[In the following M_3, $M_3{}'$, N_3, $N_3{}'$, etc., have the values given to them in § 5.]

(i) **For P [3]**, $D_2=D_1$ gives $M_3=0$, ...(7·1)

which is equivalent to

$$r/(R+d)+r/(R-d)=1. \qquad (7·2)$$

(ii) **For P [4]**, $r_2=0$ gives $M_4=0$, ...(7·3)

which is equivalent to

$$[r/(R+d)]^2+[r/(R-d)]^2=1. \qquad (7·4)$$

(iii) **For P [5]**, $D_3=D_2$ gives $M_5=0$, ...(7·5)

which is equivalent to

$$(R^2-d^2)^3+2Rr(R^2-d^2)^2-4R^2r^2(R^2-d^2)-8Rd^2r^3=0. \quad ...(7·6)$$

(iv) **For P [6]**, $r_3=0$ gives $M_6=0$. ....(7·7)

(v) **For P [7]**, $D_4=D_3$ gives $M_7=0$, ...(7·8)

which is equivalent to

$$\begin{aligned}
(R^2-d^2)^6 &-4Rr(R^2-d^2)^5-4R^2r^2(R^2-d^2)^4 \\
&+8Rr^3(R^2-d^2)^3(R^2+3d^2)-16R^2d^2r^4(R^2-d^2)^2 \\
&-32Rd^2r^5(R^4-d^4)+64R^4d^2r^6=0. \qquad (7·9)
\end{aligned}$$

(vi) **For P [8]**, $r_4=0$ gives $M_8=0$, ...(7·10)

which is equivalent to

$$\begin{aligned}
(R^2-d^2)^8 &-8r^2(R^2-d^2)^6(R^2+d^2) \\
&+8r^4(R^2-d^2)^4(R^4+d^4+10R^2d^2) \\
&-128R^2d^2r^6(R^2-d^2)^2(R^2+d^2) \\
&+128R^2d^2r^8(R^4+d^4)=0. \qquad (7·11)
\end{aligned}$$

(vii) **For P [9]**, use $D_5=D_4$;

For P [11], use $D_6=D_5$; and so on.

(viii) We can get $P\,[2m]$, $P\,[3m]$, $P\,[4m]$, etc., by replacing d and r in $P\,[m]$ by (d_2,r_2), (d_3,r_3), (d_4,r_4), etc.

For example, we can get $P\,[10]$ by replacing d and r in $N_5=0$ by d_2 and r_2 [use (5·10), (5·13) and (5·14)].

Again, **for P [9]**, we have $R^2-d_3{}^2=2Rr_3$, which is equivalent to

$$R^2N_3{}^4N_3{}'^4-d^2M_3{}^4M_3{}'^4=2RrN_3{}^2N_3{}'^2M_6. \qquad (7·12)$$

Similarly for $P\,[12]$, use $R^2-d_4{}^2=2Rr_4$; and so on.

(ix) Again we have

$$r_2/(R+d_2) = M_4/N_4', \quad r_2/(R-d_2) = M_4/N_4.$$

Hence from (7·2) and (7·4), we have

For P [6], $1/N_4 + 1/N_4' = 1/M_4.$ ....(7·13)

For P [8], $1/N_4{}^2 + 1/N_4'{}^2 = 1/M_4{}^2.$ ....(7·14)

(x) Similarly, by calculating $r_3/(R \pm d_3)$, we get from (7·2) and (7·4),

For P [9],

$$[RN_3{}^2 N_3'{}^2 + d M_3{}^2 M_3'{}^2]^{-1} + [RN_3{}^2 N_3'{}^2 - d M_3{}^2 M_3'{}^2]^{-1} = [r M_6]^{-1}. \ (7·15)$$

For P [12],

$$[RN_3{}^2 N_3'{}^2 + d M_3{}^2 M_3'{}^2]^{-2} + [RN_3{}^2 N_3'{}^2 - d M_3{}^2 M_3'{}^2]^{-2} = [r M_6]^{-2} ; \ (7·16)$$

and so on.

8. P′ [4m].

For $P'[4m]$, $D_{2m} \to \infty$, and hence

$$1 - D_m{}^4 = 0. \ ....(8·1)$$

It will be convenient to suppose that the ambiguity of the signs of R, r, d is absorbed in R, r, d, and write the condition in the form $D_m = 1$. Then

For P′ [4], $D_1 = 1$ gives $R = d.$ ....(8·2)

For P′ [8], $D_2 = 1$ gives

$$R^2 - d^2 = 2r\sqrt{(Rd)}. \ ....(8·3)$$

For P′ [12], $D_3 = 1$ gives

$$(R^2 - d^2)^2 = 4r^2\sqrt{(Rd)}\,[R + d - \sqrt{(Rd)}]. \ ....(8·4)$$

For P′ [16], $D_4 = 1$ gives

$$N_4 N_4' = 4r\sqrt{(Rd)}\,(R^2 - d^2)\,M_4. \ ....(8·5)$$

If we replace (d, r) in $P'[16]$ by (d_3, r_3), we get $P'[48]$; and so on.

9. P′ [2(2m+1)].

For $P'[2(2m+1)]$, $D_{2m+1} \to \infty$.

Hence $D_{m+1}^2 D_m^2 = 1$.

Now if $D_k{}'$ denotes the result when R and d are interchanged in the expression for D_k in terms of (R, d, r), we get $D_1{}' = \sqrt{(R/d)} = 1/D_1$; $D_2{}' = -D_2$; $D_3{}' = 1/D_3$; $D_4{}' = -D_4$; $D_5{}' = 1/D_5$; and so on.

We can prove generally that

$$D_{2m+1}' = 1/D_{2m+1}, \quad \text{and} \quad D_{2m}' = -D_{2m}.$$

Again, for $P[2m+1]$, $D_m^2 = D_{m+1}^2$; if we interchange R and d, this condition becomes $D_m^2 D_{m+1}^2 = 1$, which is the condition for $P'[2(2m+1)]$.

c

Hence we have the following rule :

For $P'[2(2m+1)]$, *interchange* R *and* d *in* $P[2m+1]$.

Professor Chaundy uses this device.

Hence we can write down $P'[6]$, $P'[10]$, $P'[14]$, etc., from $P'[3]$, $P'[5]$, $P'[7]$, etc.

For **P'[6]**, $R^2 - d^2 = 2dr$. ....(9·1)

For **P'[10]**,

$$(R^2 - d^2)^3 - 2dr(R^2 - d^2)^2 - 4d^2r^2(R^2 - d^2) + 8r^3R^2d. \ \ ......(9·2)$$

10. *The Poristic relation between two conics.*

If in the foregoing S and S' are conics, then S_2, S_3, S_4, etc., of § 3 are also conics of the pencil $S' + V_t\,S = 0$, where $V_1 = 0$, and V_2, V_3, ... , V_{n-1} are quantities which can be determined by a suitable *Recurrence Formula*.

For porism, $V_{n-1} = V_1 = 0$, or the equivalent condition, $V_n \rightarrow \infty$.

To find the necessary recurrence formula, we use the following theorem due to Salmon (*Conic Sections*, page 343, Ex. 3) :

If a triangle is inscribed in a conic S such that its sides touch the conics $S' + V_m\,S = 0$, $S' + V_k\,S = 0$, $S' + V_t\,S = 0$, then

$$(\Theta' - \triangle \Sigma\, V_m V_k)^2$$
$$- 4(\triangle' + \triangle\, V_m V_k V_t)(\Theta + \triangle \Sigma\, V_m) = 0, \ \ ....(10·1)$$

where

$$\triangle k^3 + \Theta k^2 + \Theta' k + \triangle' = 0$$

is the discriminant of $kS + S'$.

Now when the polygon $A_1 A_2 \dots A_n$ is inscribed in S and circumscribed to S', we see, as in § 4, that triangles can be inscribed in S such that if two of its sides touch $S' + V_m S = 0$ and $S' + V_k S = 0$, then the third side touches either $S' + V_{m+k} S = 0$, or $S' + V_{m-k} S = 0$. Hence V_{m+k} and V_{m-k} are the two values of V_t given by the equation (9·1).

Hence, when $m > k$,

$$\triangle^2 (V_m - V_k)^2 V_{m+k} V_{m-k}$$
$$= (\Theta' - \triangle\, V_m V_k)^2 - 4\triangle'\, [\Theta + \triangle\,(V_m + V_k)]. \ \ ......(10·2)$$

Since $V_1 = 0$, this gives

$$\triangle^2 V_m{}^2 V_{m+1} V_{m-1} = \Theta'^2 - 4\triangle'\Theta - 4\triangle\triangle'\, V_m. \ \ ....(10·3)$$

Again, the sum of the roots of the equation (9·1) for V_t gives, when $k = 1$,

$$V_{m+1} + V_{m-1} = 2(\Theta' V_m + 2\triangle')/\triangle\, V_m{}^2. \ \ ....(10·4)$$

Now, when $V_m = V_1$ and $V_k = V_1$, we have $V_t = V_2$. Hence writing $V_m = 0$, $V_k = 0$, $V_t = V_2$ in (9·1), we get

$$4\triangle\triangle' V_2 = \Theta'^2 - 4\triangle'\Theta. \ \ ....(10·5)$$

Again, putting $m = 2$ in (9·4), we have

$$\triangle V_2{}^2 V_3 = 2(\Theta' V_2 + 2\triangle'),\cdot$$

hence $\qquad\qquad P_3{}^2 V_3 = 8\triangle'P_4,$ ....(10·6)

where $\qquad\qquad P_3 = \Theta'^2 - 4\triangle'\Theta,$

and $\qquad\qquad P_4 = \Theta'^3 - 4\triangle'\Theta\Theta' + 8\triangle\triangle'^2.$

We thus see that (10·3) *is a Recurrence Formula for* V_3, V_4, V_5, *etc.*, where V_2 and V_3 are given by (10·5) and (10·6).

For the poristic condition, call it $\pi[n]$ of n sides, $V_n \to \infty$, or its equivalent $V_{n-1} = 0$.

For $\pi[m+k]$, $V_{m+k} \to \infty$, or its equivalent $V_m = V_k$.

[*Note :* The case of two circles can be deduced from the results of this section ; but the text was preferred so as to base the discussion entirely on elementary principles. It may be mentioned here that the possibility of deriving a recurrence formula for V_k with the aid of Salmon's theorem was pointed out by me in 1918 in the *Journal of the Indian Mathematical Society* (Vol. X, page 475).]

11. *To find* V_4, V_5, *etc.*

We have $\quad \triangle^2 V_2 V_3{}^2 V_4 = P_3 - 4\triangle\triangle'V_3,$

hence $\qquad 16\triangle\triangle'P_4 V_4 = P_3\,[P_3{}^3 - 32\triangle\triangle'^2 P_4].$ ....(11·1)

Again, $\qquad \triangle^2 V_3 V_4{}^2 V_5 = P_3 - 4\triangle\triangle'V_4,$

which gives $\qquad P_5{}^2 V_5 = 8\triangle'P_3 P_4 P_6,$ ....(11·2)

where $\qquad\qquad P_5 = P_3{}^3 - 32\triangle\triangle'^2 P_4,$

and $\qquad\qquad P_6 = 4P_4{}^2 - P_3{}^3 + 32\triangle\triangle'^2 P_4.$

Similarly for V^6, V^7, etc.

12. *To find* $\pi[3]$, $\pi[4]$, $\pi[5]$, *etc.*

For $\pi[3]$, $\quad V_2 = 0$ gives $P_3 = 0$. ....(12·1)

For $\pi[4]$, $\quad V_3 = 0$ gives $P_4 = 0$. ....(12·2)

For $\pi[5]$, $\quad V_3 = V_2$ gives $P_5 = 0$. ....(12·3)

For $\pi[6]$, $\quad V_4 = V_2$ gives $P_6 = 0$. ....(12·4)

For $\pi[7]$, $\quad V_4 = V_3$ gives

$\qquad\qquad P_3{}^6 - 32\triangle\triangle'^2 P_3{}^3 P_4 = 128\triangle\triangle'^2 P_4{}^3.$ ....(12·5)

For $\pi[8]$, $\quad V_5 = V_3$ gives $P_3{}^3 P_6 = P_5{}^2.$ ....(12·6)

For $\pi[9]$, $\quad V_5 = V_4$ gives $128\triangle\triangle'^2 P_4{}^2 P_6 = P_5{}^3.$

For $\pi[11]$, $\quad V_6 = V_5.$

Hence $\qquad\qquad \triangle^2 V_4 V_5{}^3 = P_3 - 4\triangle\triangle'V_5\,;$

and so on.

University College, Colombo. $\qquad$ F. H. V. GULASEKHARAM.

GLEANINGS: AN APPEAL.

The Editor will be grateful for help in the filling up of odd corners. A precise reference should accompany every quotation.

CORRESPONDENCE.

LONG MULTIPLICATION OF MONEY.

To the Editor of the *Mathematical Gazette.*

DEAR SIR,—On p. 256 of your October number Mr. Webb gives a long multiplication of money sum done by what he follows Dr. Ballard in calling the " Wholesale Method ". I should like to draw your readers' attention to an arrangement which differs only slightly from that given by Mr. Webb, but which seems to me to have distinct advantages.

Taking the sum he gives, I start thus :

£	s.	d.	f.
78	16	9	3
			249
17430	2490	2241	747
1992	1494		

I lay great stress on :

(i) The 249 is multiplied by the numbers of £, s., d., and farthings.

(ii) This multiplication is done all at once before any transference from column to column is made.

Then the 747 farthings are changed to 186d. and 3 farthings, and these pence are transferred to the pence column as indicated by the arrow (see below), and so on. The whole sum appears thus :

£	s.	d.	f.
78	16	9	3
			249
17430	2490	2241	4) 747
1992	1494	186 ←	186d. 3f.
209 ←	202 ←	12) 2427	
£19631	20) 4186	202s. 3d.	
	£209 6s.		

Product = £19631 6s. 3¾d.

When the method becomes familiar the arrows may be omitted.

The fact that all the multiplication is done at one time seems to me a distinct advantage.

I have placed the multiplier (249) below the sum of money as I have always done, but I think Mr. Webb's idea of placing the multiplier above the sum of money is probably better.

Perhaps Mr. Webb will have opportunity of trying out my arrangement against the one he gives.

It is not without interest that Dr. Ballard and I, working quite independently of one another, both arrived at arrangements which

are nearly identical. I have used the above arrangement regularly since 1924, Dr. Ballard gave his in his *Teaching the Essentials of Arithmetic* published in 1928 ; my arrangement first appeared in print in Godfrey and Siddons' *Teaching of Elementary Mathematics* in 1931. Sir Percy Nunn's arrangement for long division of money certainly suggested the idea to me and probably also to Dr. Ballard.

I doubt whether " Wholesale Method ", as used by Dr. Ballard and Mr. Webb, is a good description of the method ; it has been described as the method in which the numbers of £ s. d. are used as multipliers—an excellent description, though too long for a name.

Yours truly, A. W. SIDDONS.

To the Editor of the *Mathematical Gazette*.

DEAR SIR,—I have seen a copy of Mr. Siddons' letter to you concerning his method for long multiplication of money.

I was at once attracted to this method and felt it worth while carrying out what tests I could in accordance with Mr. Siddons' suggestion.

Unfortunately I was unable to make the tests as extensive as I desired, but the report of what I have managed to do may be of interest to your readers.

Yours truly,

H. WEBB.

[Mr. Webb's report, in which he analyses the results of these further tests, will appear in the May issue of the *Gazette*.—ED.]

1340. For instance, many people believe that some families are more liable than others to run exclusively to boys or to girls. Yet an analysis of 53,680 families, each having eight children, showed that 557 were all boys or all girls. By mathematically pure chance the numbers would have been 419, which is in good agreement.—Review of *You and Heredity*, by A. Scheinfield, in *The Manchester Guardian Weekly*, 15th September, 1939. [Per Mr. R. A. Fairthorne.]

1341. Condorcet the mathematician solved a mathematical problem which had worried him the day previous, during a somnambulistic trance, and had no recollection of the fact the following morning.—E. C. Spitzka, *Insanity*, Pt. I, Ch. 6, p. 58 ; quotation from the *Century Dictionary*. [Per Mr. E. B. Escott.]

1342. HOW MANY IS " MOST " ?
Reno, the city of divorce and love tangles, claims that it is entitled, by merit, to be called " The Gretna Green of the West ". Despite its publicity about divorces, marriages there exceed divorces by six to one. Most of the marrying people there, adds the B.U.P., are those who have just been divorced. —*Evening News*, 6th January, 1940. [Per Professor L. M. Milne-Thomson.]

1343. Did one say quantities of gooseberries or numbers? The only thing his mother, so far as he could remember, had ever taught him, during the period he was at her knee, was not to say quantities if what he meant was numbers.—*The Jasmine Farm*, p. 9, by the author of *Elizabeth and her German Garden*. [Per Mr. A. R. Miller.]

MATHEMATICAL NOTES.

1495. *Road and railway curves.*

It is a common exercise on circular motion to calculate the necessary " super-elevation ", E, of the outer rail of a railway curve, assumed to be a circular arc. For the standard British gauge $E = 3 \cdot 76 v^2 / R$ inches if v is in miles per hour and R in feet. If κ is the curvature, $E = 3 \cdot 76 v^2 \kappa$. I understand that on London's Underground system the minimum radius is 20 chains and allowance is made for speeds of 40 miles per hour (though this is of course exceeded), which gives $E = 4\frac{1}{2}$ inches. It is obvious that the circular arc alone is impossible as a railway curve, for there would be an abrupt rise or fall of several inches at the points where the arc joins or leaves the straight. The elevation being proportional to the curvature of the track, it is necessary to use as a " transition " from the straight to the circular arc a curve whose curvature increases uniformly from zero to that of the circular arc. More precisely, if s is the length of the arc of the transitional curve measured from its point of tangency with the straight,

$$\kappa = d\psi/ds = as \; ;$$

this gives
$$\psi = \tfrac{1}{2}as^2,$$

the intrinsic equation of Glover's spiral, much used in road curves. Other curves used are the lemniscate $r^2 = a^2 \sin 2\theta$ and the cubical parabola, $y = ax^3$; these only approximately satisfy $\kappa = as$. The cubical parabola, for instance, gives near the origin

$$\kappa \simeq d^2y/dx^2 = 3ax \simeq 3as,$$

but it is commonly used on railways in the following manner. Let
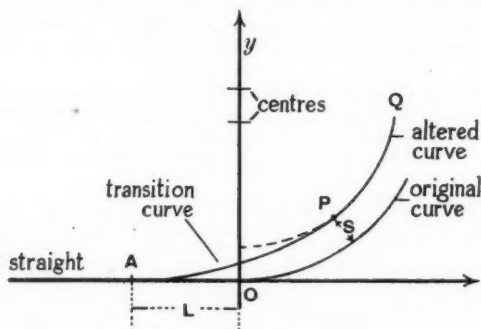
Fig. 1.

the transition curve be $y = a(x + L)^3$ referred to the tangent and radius at O. Since R is large compared with the range of x con-

sidered, we may take the equation of the original (circular) curve to be $y = x^2/2R$. The "altered" circular curve will then be

$$y = (x^2/2R) + S,$$

where S is the lateral "shift" of the track.

If the transition curve runs into the shifted curve at P, we must have y, dy/dx and κ the same for both curves at P. Since dy/dx is small, we may safely take $\kappa = d^2y/dx^2$.

Then   (i) $a(x+L)^3 = (x^2/2R) + S$;

                  (ii) $3a(x+L)^2 = x/R$;   (slopes)

                  (iii) $6a(x+L) = 1/R$;   (curvatures).

From (ii) and (iii), $x = L$;

from (iii) $a = 1/12RL$

and P is $(L, 2L^2/3R)$,

and from (i) $S = L^2/6R$

and P is $(L, 4S)$.

The railway engineer takes $L = 300E$, to give a reasonable gradient on the outer rail. He can then calculate S and the necessary offsets from Ox.

In the case of road curves, which are usually of much greater curvature than rail curves, it is just as important to design a satisfactory transition. For the curvature of the path of a car is roughly proportional to the angle through which the steering wheel has been turned from its normal position. If the curve at a right-angled bend is a circular quadrant, then the curvature is a discontinuous function of the distance (Fig. 2) and the wheel would have to be given an
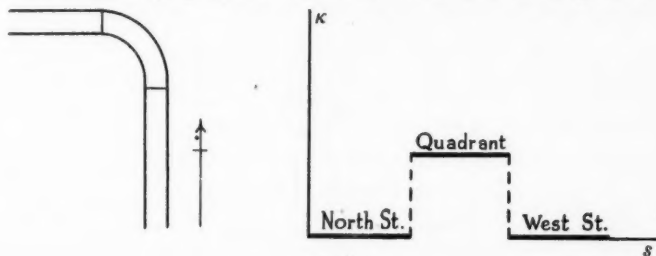
FIG. 2.

instantaneous wrench. For driving comfort, the rate of turning the wheel should be constant, or have only a small angular acceleration, that is, $d^2\kappa/ds^2$ should be as small as possible, implying $\kappa \simeq as$. It follows that a road curve whose "transitions" are arcs of Glover's spiral and which is also correctly "banked" will be extremely comfortable for both passenger and driver.

The Programme Committee might well invite a Civil Engineer to give a talk on this subject at some future Annual Meeting.

<div align="right">A. P. ROLLETT.</div>

1496. *On finding complex roots by iteration.*

The method of iteration for solving the equation $x = \phi(x)$ is usually justified by an appeal to the theorem of Mean Value,

$$\phi(a) - \phi(\xi) = (a - \xi) \cdot \phi'[\xi + \theta(a - \xi)].$$

The proof therefore applies to real roots only. Complex roots can be brought within the scope of the method by the following theorem :

Let ζ be a root of the equation $z = \phi(z)$, a an approximation. Then, if $\phi(z)$ is continuous, $\phi'(z)$ integrable and $|\phi'(z)| < 1$ on the straight line from ζ and a, $\phi(a)$ is a closer approximation than a.

Proof. Since $\phi(z)$ is continuous,

$$\phi(a) - \phi(\zeta) = \int_\zeta^a \phi'(z)\, dz.$$

The modulus of the integral does not exceed ML where L is the length of the contour, M the greatest value of $|\phi'(z)|$ on it. Taking as contour the straight line from ζ to a, $L = |a - \zeta|$ and as $\phi(\zeta) = \zeta$ and $M < 1$,

$$|\phi(a) - \zeta| < |a - \zeta|,$$

the required result.

The proposition is useful for finding the large proper values of certain boundary problems for which Green's function is unsymmetric, and the proper values therefore not necessarily real.

The proper values of one boundary problem are the squares of the roots of the equation

$$\cos 2z = \cosh 2\beta - (k \sin 2z)/z, \quad\text{.........................(i)}$$

where k and β are real, $\beta \neq 0$. This becomes $z = \phi(z)$ where

$$2\phi(z) = \text{arc cos } \{\cosh 2\beta - (k \sin 2z)/z\},$$

so that

$$2\phi'(z) = \frac{k}{z} \cdot \frac{2 \cos 2z - (\sin 2z)/z}{\sqrt{[1 - \{\cosh 2\beta - (k \sin 2z)/z\}^2]}}.$$

Now, if $z = x + iy$,

$$\left.\begin{array}{l} \sin z = \sin x \cosh y + i \cos x \sinh y, \\ \cos z = \cos x \cosh y - i \sin x \sinh y, \end{array}\right\} \quad\text{...............(ii)}$$

Hence if y is bounded, $|\phi'(z)| < 1$ whenever $|z|$ is large enough ; that is, iteration is permissible if y is bounded near large roots. To prove that this is so, we re-write (i) in the form

$$\sin (z + i\beta) \sin (z - i\beta) = (k \sin z \cos z)/z,$$

and take the modulus of both sides. From (ii)

$$|\sin z| = \sqrt{(\cosh^2 y - \cos^2 x)}, \quad |\cos z| = \sqrt{(\cosh^2 y - \sin^2 x)},$$

which lead to

$$\sinh^2 \beta\, (\coth^2 \beta - \tanh^2 y)\sqrt{[1 - \cos^2 x/\cosh^2 (y + \beta)]}$$
$$\times \sqrt{[1 - \cos^2 x/\cosh^2 (y - \beta)]}$$
$$= |k/z| \cdot \sqrt{(1 - \cos^2 x/\cosh^2 y)} \cdot \sqrt{(1 - \sin^2 x/\cosh^2 y)}.$$

Since $\sinh^2 \beta$ and $\coth^2 \beta - \tanh^2 y$ do not vanish and the square roots on the right are bounded for all values of x and y (which are real), it follows that when $|z|$ is large, one of the square roots on the left must be small. Suppose to begin with that the second of these is small. This evidently cannot happen unless $\cos^2 x \cong 1$, $\cosh^2 (y - \beta) \cong 1$, that is, $x \cong n\pi$, $y \cong \beta$, and then the other square root on the left is not small (since $\beta \neq 0$), from which it follows that

$$1 - \cos^2 x/\cosh^2 (y - \beta) \text{ is } O(1/z^2).$$

Put $x = n\pi + \epsilon$, $y = \beta + \delta$, where the integer n is so chosen that $-\frac{1}{2}\pi < \epsilon \leqslant \frac{1}{2}\pi$. Then

$$1 - \cos^2 x/\cosh^2 (y - \beta) = 1 - \cos^2 \epsilon/\cosh^2 \delta$$
$$= \tanh^2 \delta + \sin^2 \epsilon/\cosh^2 \delta.$$

Accordingly, $\tanh^2 \delta$ and $\sin^2 \epsilon/\cosh^2 \delta$ are each $O(1/z^2)$, so that δ and ϵ are each $O(1/z)$, and therefore $y = \beta + O(1/z)$ near any large root z. This establishes that iteration is permissible for large roots.

Taking $z = n\pi + i\beta$ as the first approximation, the second is obtained by substituting this value for z in the right-hand side of equation (i) :

$$\cos 2z = \cos 2i\beta - \frac{k \sin 2i\beta}{n\pi + i\beta}$$
$$= \cos \left(2i\beta + \frac{k}{n\pi}\right) + O(1/n^2).$$

Neglecting terms in $1/n^2$, the second approximation is

$$z = n\pi + i\beta + (k/2n\pi).$$

All other large roots of the equation correspond to small values of $1 - \cos^2 x/\cosh^2 (y + \beta)$, and therefore all large roots of equation (i) are given by

$$z = n\pi + (k/2n\pi) \pm i\beta + O(1/n^2).$$

<div align="right">G. B. EHRENBORG.</div>

1497. *On some applications of the (so-called) Lodge's Theorem on the motion of the centre of gravity.* (Note 1465.)

This theorem is known to the nautically-minded as " the shifting of Deck-Cargo Theorem ", because a storm may shift a deck-cargo across the deck and alter the position of the centre of gravity of a ship and cargo. It has a useful application in the calculation of a class of centroids, as pointed out by Professor J. G. Gray in 1922 (*Phil. Mag.*, 6th Series, vol. 44, p. 247).

In the figure (p. 42), O may be a point or represent an axis of revolution ; A may trace out a circular arc, OA a circular sector, or OA may be the projection of an area tracing out a sector of a volume of revolution. The length, area or volume traced out varies directly as the angle of revolution.

Let the object (arc or sector of a circle, of a sphere, cone, etc.) whose centroid is required, be represented by AOB, where $\angle AOB = \alpha$, and let the mass be $m\alpha$. The mass is now revolved through a further angle α/n, so as to take up a position represented by $A'OB'$. In the process the centroid moves with the bisector of $\angle AOB$ from G to G', where $\angle GOG' = \alpha/n$. If $OG = R$, then $GG' = 2R \sin(\alpha/2n)$. But this is tantamount to transferring the sector AOA' to the position BOB', that is to say, to shifting a deck-cargo of mass $m\alpha/n$ from H to H'. Evidently $\angle HOH' = \alpha$, so that if OH is r,

$$HH' = 2r \sin \tfrac{1}{2}\alpha.$$
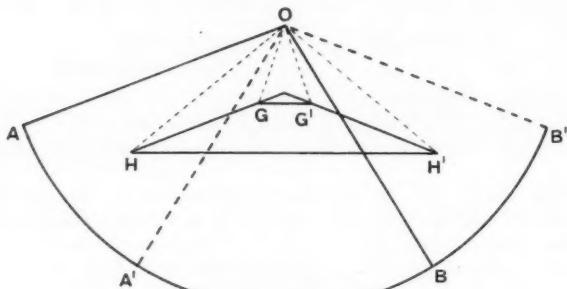
Fig. 1.

We now have $HH' = n \cdot GG'$, by Lodge's rule, so that

$$2r \sin(\alpha/2) = 2nR \sin(\alpha/2n).$$

It follows that

$$\frac{r}{R} = \frac{n \sin(\alpha/2n)}{\sin(\alpha/2)} = \frac{\sin(\alpha/2n)}{\alpha/2n} \Big/ \frac{\sin(\alpha/2)}{\alpha/2}.$$

Hence the distance of the centroid of any sector from O is given by $\kappa (\sin \tfrac{1}{2}\alpha/\tfrac{1}{2}\alpha)$, where κ depends on the shape of what is revolved. All we have to do now is to ascertain the distance of the centroid from O for some given value of α. For instance, we may let n tend to infinity and the result becomes $R = R_0 \cdot (\sin \tfrac{1}{2}\alpha/\tfrac{1}{2}\alpha)$, where in the case of a circular arc $R_0 = a$, the radius of the circle, and in the case of a circular sector $R_0 = 2a/3$ as the sector becomes indistinguishable from a triangle in the limit.

For the centroid of the surface of the lune of a sphere, we use the fact that the centroid of a hemispherical bowl is at a distance $\tfrac{1}{2}a$ from the centre, deducing this, of course, from the equality of areas of zones cut off by equidistant parallel planes. Putting $\alpha = \pi$, we get $\tfrac{1}{2}\alpha = \kappa/\tfrac{1}{2}\pi$ so that $R = \tfrac{1}{4}\pi a (\sin \tfrac{1}{2}\alpha/\tfrac{1}{2}\alpha)$. For the centroid of the corresponding volume, we use the position of the centroid of a solid hemisphere in the same way, so that in this case $R = \tfrac{3}{16}\pi a (\sin \tfrac{1}{2}\alpha/\tfrac{1}{2}\alpha)$. For solids of revolution a little thought shows that the limiting distance R_0 is the depth of the centre of pressure of the area tracing

out the solid on the assumptions that the area is vertical, the axis of revolution is in the surface of the water, and atmospheric pressure is disregarded. Hence, for instance, for a sector of a right circular cone of base radius a, $R_0 = \frac{1}{2}a$. On the other hand, the centre of pressure of a vertical semicircle whose diameter is in the surface is deduced to be at a depth $3\pi a/16$ below the surface.

In general $R_0 = K^2/d$, where K is the radius of gyration of the area about the axis of revolution and d the distance of its centroid from the same axis. For instance, for a circle of radius a whose centre is at a distance d from a straight line in its plane, $K^2 = d^2 + \frac{1}{4}a^2$. If the plane now revolves about the straight line through an angle α, the circle traces out a portion of an anchor ring, whose centroid is accordingly at a distance from the axis given by the formula

$$\frac{d^2 + \frac{1}{4}a^2}{d} \cdot \frac{\sin \frac{1}{2}\alpha}{\frac{1}{2}\alpha}.$$
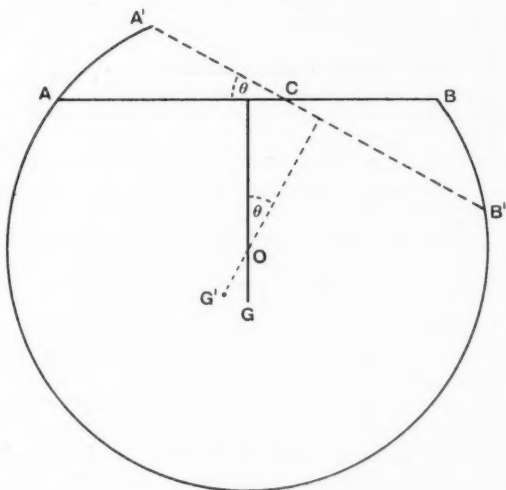
Fig. 2.

Segments of cylinders and spheres may be somewhat similarly treated. In Fig. 2 the segment is rotated about O through an angle θ. The deck-cargo is now the wedge whose projection on the plane of the paper is ACA', which is shifted to BCB'. It must here be assumed that θ is small, so that any element of area dS in the plane section represented by AB moves through an arc $x\theta$ in its rotation, where x is the distance of the element from C.

In this way a volume $x\theta \cdot dS$ is moved a distance practically equal to $2x$ to a symmetrical position on CB'. The deck-cargo being thus shifted piecemeal, apply the rule in the form " mass of deck-

cargo × shift of deck-cargo = mass of ship and cargo × shift of its centre of gravity ". Then

$$\Sigma \rho x \theta . dS . 2x = \rho V . GG',$$

where V is the volume, and ρ the density of the segment, and the summation is over the portion of the top represented by CA. Now $GG' = 2OG \sin \frac{1}{2}\theta$, so that

$$\Sigma \rho . 2x^2 . dS = \rho V . 2OG . (\sin \frac{1}{2}\theta)/\theta.$$

In the limit $\Sigma 2x^2 . dS$ taken over the portion represented by CA is equal to $\Sigma x^2 dS$ taken over the whole of the top, whether rectangle or small circle, while $2(\sin \frac{1}{2}\theta)/\theta \to 1$. Hence with an obvious notation $OG = AK^2/V$, in all these cases.

A precisely similar argument would show that in the case of a segment of a circular lamina,

$$OG = (\tfrac{2}{3}a^3)/(\text{area of segment})$$

where $2a$ is the length of the chord. C. L. WISEMAN.

1498. *The Parallelogram Law for Vectors.*

The following discussion is expressed in terms of forces acting along intersecting straight lines, but is intended to apply to any class of directed quantities which satisfy the given assumptions. The Transmissibility of Force is not assumed, as in Duchayla's proof.

1. Forces P, Q acting along lines OA, OB are equivalent to a force R acting along a line OC, and conversely. R is called the resultant of P, Q. The term " equivalent " is used in a physical sense : it implies having identical quantitative effects, but the argument seems to be valid when some kinds of effect are ignored. Thus " statically equivalent force systems " acting on a rigid body may set up different stresses ; the discussion may be applied to forces acting along skew lines if it is understood that the equivalence does not extend to couple effects.

2. P, Q have no order of precedence : the resultant of Q, P is identical with that of P, Q.

3. Where OA, OB have the same direction, $R = P + Q$, the letters standing for the magnitudes of the forces. The case of " unlike " forces follows at once if P is replaced by $P - Q$ and Q, and it is further assumed that the Q's in opposite senses annul each other. Of course OC has the same direction as OA, if $P > Q$.

4. $R = 0$ if, and only if, P, Q annul each other (or are in equilibrium), and this only occurs when P, Q are equal in magnitude and act in opposite directions. This implies that two forces cannot be equivalent unless identical in magnitude and direction, and it follows that the directions OA, OB, OC are either all parallel to the same line or all different. Also, given the lines OA, OB, OC, the resolution of R into P, Q is unique. For if P', Q' are alternative components, the pair P, Q is equivalent to the pair P', Q'. Let the pair P' reversed, Q' reversed be superimposed on each of the pairs.

Then the forces $P - P'$ along OA, $Q - Q'$ along OB are equivalent to four forces in equilibrium, which is contrary to the assumption unless $P = P'$, $Q = Q'$.

5. The proof only requires to use the case where the angle BOA is a right angle, but the further assumption is made that the lines OA, OB, OC are in one plane. Euclidean space is assumed.

6. An assumption of continuity is made for P, Q as the angles made with R vary. The proof is then as follows :

R along OC is equivalent to X along OA and Y along OB, the angles AOC, COB being θ, $\pi/2 - \theta$ and the lines in one plane. By superimposition, nR is equivalent to nX, nY, the lines of action being unchanged and n being first any positive integer and by the usual extension any real number. As $nX/nR = X/R$, it follows that X/R for any given θ is not a function of R but we may write $X = RC(\theta)$, where $C(\theta)$ is a function of θ whose properties must be examined. Clearly we can write $Y = RS(\theta)$ where $S(\theta)$ stands for $\cdot C(\pi/2 - \theta)$ and $C(0) = 1$, $C(\pi/2) = 0$. So long as θ does not lie outside the range 0, $\pi/2$, $C(\theta)$ cannot be zero except when $\theta = \pi/2$, and $S(\theta)$ cannot be zero except when $\theta = 0$. Otherwise we should have forces R and Y (or R and X) equivalent although having different directions. Hence by Assumption 6, the C and S functions are positive within the range.

Let OE, OF be the positions taken by OA, OB after each has been rotated through an angle α in their own plane. If $\theta > \alpha$, R is equivalent to $RC(\theta - \alpha)$ along OE and $RS(\theta - \alpha)$ along OF. But the components $RC(\theta)$, $RS(\theta)$ along OA, OB give components along
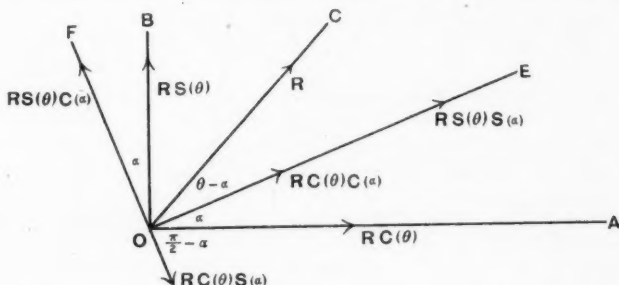
OE, OF, as shown in the figure. As the resolution along OE, OF is unique

$$C(\theta - \alpha) = C(\theta) C(\alpha) + S(\theta) S(\alpha),$$
$$S(\theta - \alpha) = S(\theta) C(\alpha) - C(\theta) S(\alpha).$$

Hence $\quad C(\theta - \alpha) + iS(\theta - \alpha) = [C(\theta) + iS(\theta)] [C(\alpha) - iS(\alpha)]$

where $i = \sqrt{(-1)}$; and

$$[C(\theta - \alpha) + iS(\theta - \alpha)] [C(a) + iS(\alpha)] = [C(\theta) + iS(\theta)] [C^2(\alpha) + S^2(\alpha)].$$

The factor $C^2(\alpha) + S^2(\alpha) = 1$, since it is equal to $C(\alpha - \alpha) = C(0)$.

The relation $[C(\theta - \alpha) + iS(\theta - \alpha)][C(\alpha) + iS(\alpha)] = C(\theta) + iS(\theta)$ being true for all values of θ, α such that $0 < \alpha < \theta < \pi/2$, it follows that within this range $C(\theta) + iS(\theta) = e^{(a+ib)\theta}$, a and b being real constants, whence $C(\theta) = e^{a\theta}\cos b\theta$, $S(\theta) = e^{a\theta}\sin b\theta$.

Since $C^2(\theta) + S^2(\theta) = 1$, $a = 0$.

Since $C(\theta)$ changes continuously from 1 to 0 and $S(\theta)$ from 0 to 1, neither function being zero for intermediate values of θ within the range 0, $\pi/2$, b must be 1. (Values numerically < 1, values numerically > 1 and the value -1 are all seen to be excluded.)

Hence the rectangular components of R are $R\cos\theta$, $R\sin\theta$, and the general form of the Parallelogram Law is easily deduced.

<div align="right">H. W. UNTHANK.</div>

1499. *Eighteen parallelograms.*

Curiosity about the eighteen parallelograms of Mr. Gibbins' article (*Gazette*, July 1940, No. 260, p. 165) has led to the following notes :

(*a*) If two transversals cut the sides of the triangle ABC at X, Y, Z and X_1, Y_1, Z_1 respectively, and if

$$AZ_1 = ZB \quad \text{and} \quad AY_1 = YC,$$

then
$$X_1B = CX.$$
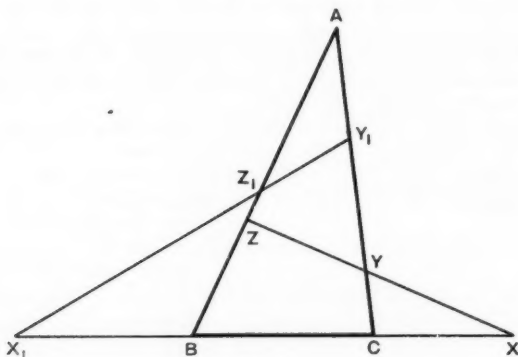
FIG. 1.

This is a simple application of Menelaus' theorem, and gives an easy proof of the collinearity of K', G', L'. (§ 3 of Mr. Gibbins' article.)

(*b*) In § 2, $GKK'G'$ is shown to be a parallelogram ; but the method used is not so easily applied to many of the eighteen parallelograms. An alternative method is suggested, which not only proves that GK and $G'K'$ are parallel to the line PQR but shows also that $G'K' = \frac{1}{2}PQ$, $G'L' = \frac{1}{2}QR$, and further gives a simple method of showing that P, Q, R lie on a straight line.

In the figure, X, Y are the mid-points of DC, EB. Further, $EG' = DA$ and is therefore parallel to XP and equal to $2XP$; $EK' = CB$ and is therefore parallel to XQ and equal to $2XQ$.
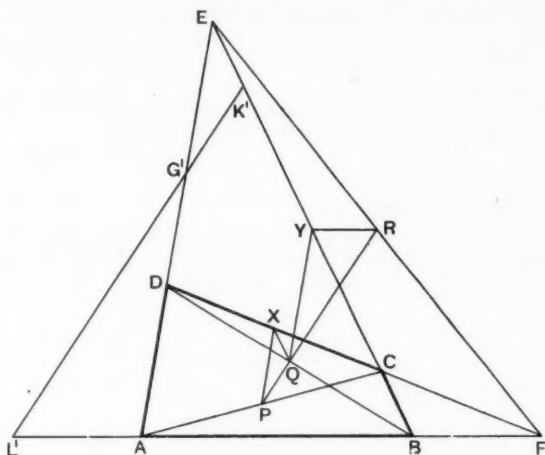

Fig. 2.

Hence $G'K'$ is parallel to PQ and equal to $2PQ$. Again, $ED = G'A$ and is parallel to YQ and equal to $2YQ$; $AL' = BF$ and is parallel to YR and equal to $2YR$. Hence $G'L'$ is parallel to QR and equal to $2QR$. But $K'G'L'$ is a straight line, by (a). Hence PQR is a straight line.

(c) Menelaus' theorem shows the collinearity of the four sets of three points, and the parallelogram fact follows.

(d) The fact that KK', GG'', and another line are parallel to the line joining the mid-points of BE and DF suggests that the five other sets of lines, forming sides of the other fifteen parallelograms, have corresponding alliance with other lines associated with the given quadrilateral. Readers may find the search for this correspondence of some interest.   F. MAYOR.

1500. *"Co-central" Circles.*

If a co-axal system of circles is rotated in its plane about its centre C, a doubly infinite system of circles is obtained such that the power of C with respect to all the circles of the system is the same. C is therefore the radical centre of any three of the circles, and the system may appropriately be called a " co-central " system.

The co-central system defined by three circles $S = 0$, $S' = 0$, $S'' = 0$ is $S + \lambda S' + \mu S'' = 0$. The simplest equation to a co-central system is $x^2 + y^2 + 2\lambda x + 2\mu y + c = 0$.

In three dimensions a system of spheres having a common radical *plane* is called a co-*axal* system of spheres. This is unfortunate, for if we rotate the system about an axis through its centre we have a system having a common radical *axis*, for which the name co-axal should be reserved. By rotating the first system about its centre in all directions we have a third system with a common radical centre which we may call a co-central system of spheres. The system having a common radical *plane* should evidently be called a co-*planal* system. Is it too much to expect that this correction will be adopted in future editions of the textbooks?

D. V. A. S. Amarasehara.

1501. *A geometrical note.*

It is sometimes possible to deduce interesting results for three dimensions from two-dimensional figures by regarding the latter as " pictures " of three-dimensional configurations.
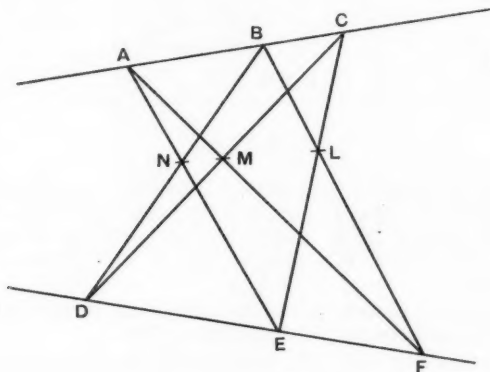
For instance, in the figure of the two-dimensional form of Pappus' theorem we may regard ABC, DEF as the picture of two *skew* lines. The fact that L, M, N are collinear indicates that the three transversals from the eye (and therefore from a general point) to the pairs of lines (BF, CE), (CD, AF), (AE, BD) are co-planar. Conversely, if any plane cuts BF, CE, CD, AF, AE, BD in L, L', M, M', N, N', then by viewing the figure from the point of intersection of MM', NN', we see that LL', MM', NN' are concurrent.

Again, if A, B, C, D, E, F are *any* six points, then by viewing them from a point on the twisted cubic through them they seem disposed on a conic, and therefore the transversals from a point on the twisted cubic to (BF, CE), (CD, AF), (AE, BD) are again co-planar.

D. V. A. S. Amarasehara.

1502. *An awkward integral*.

Is there any reasonably simple way of calculating $\int_0^a x^x\,dx$ for all positive values of a?

When $a = 1$ it is known that the integral

$$= 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \dots = 0.783430\dots.$$

This is a special case of the formula

$$\int_0^1 x^{ax^c}\,dx = 1 - \frac{a}{(c+1)^2} + \frac{a^2}{(2c+1)^3} - \frac{a^3}{(3c+1)^4} + \dots,$$

but similar methods for other values of the upper limit seem to lead to very complicated results.

I have not been able to find any rapidly convergent series either for small or large values of a. For a small I have used

$$I = \int_0^a x^x\,dx = a^{a+1} - \int_0^a x^{x+1}(1 + \log x)\,dx$$

and Simpson's rule for this integral.

For a and b large we have

$$\int_a^b x^x\,dx = \left| \frac{x^x}{1 + \log x} \right|_a^b + \int_a^b \frac{x^{x-1}}{(1 + \log x)^2}\,dx.$$

Using Simpson's rule I find 99·63 for $a = 3$, $b = 4$, but there seems to be no easy way of estimating the accuracy of the rule when applied to a function which increases so ferociously. I should be interested to hear whether anyone has a more practicable method, and give some of my results for comparison :

a	I	a	I	a	I
0	0·0000	1·2	1·0063	2·4	5·1560
0·2	0·1625	1·4	1·2887	2·6	7·1459
0·4	0·3027	1·6	1·6577	2·8	10·0893
0·6	0·4446	1·8	2·1531	3·0	14·515
0·8	0·6009	2·0	2·8337	4·0	114·1
1·0	0·7834	2·2	3·7891	5·0	1242·

(Last figure in each case uncertain.)

In the course of this work I came across a curiosity. If you ask a class for the value of $2^{\left(\frac{5}{2}\right)^{\frac{1}{4}}}$ you will probably be told that it is equal to 2. But it comes to e. G. W. BREWSTER.

1503. *The Measuring Problem*.

Two vessels A and B hold a and b pints respectively, a being prime to and less than b. To find the number of operations required to measure an assigned integral number of pints by means of these

D

vessels. Starting with A empty and B full, all quantities up to b may be obtained in the vessel B by repeated application of the transformations :

(1) Initial Position : A empty, contents of B greater than a.
   Fill A from B, empty A.
(2) Initial Position : A empty, contents of B less than a.
   Transfer contents of B to A, fill B, fill A from B, empty A.

These transformations are equivalent to the substitution S :

$$\begin{matrix} 1 & 2 & \dots & r & \dots & a & a+1 & \dots & a+t & \dots & b \\ b-a+1 & b-a+2 & \dots & b-a+r & \dots & b & 1 & \dots & t & \dots & b-a \end{matrix}$$

and determine, in order, the quantities $b, Sb, S^2b, \dots S^{b-2}b, S^{b-1}b = a$.

Since transformation (1) contains two, and transformation (2) four operations, two operations are required to pass from b to Sb, and two or four operations to pass from S^rb to $S^{r+1}b$, according as S^rb is greater or less than a. However, the last operation of each transformation consists only in emptying A, and so we must subtract unity from the total of operations obtained in this way.

Alternatively, starting with A full and B empty, and then emptying contents of A into B, we may obtain the quantities b, Sb, $S^2b, \dots S^{b-1}b$, in reverse order, by repeated application of the transformations :

(1) Initial Position : A empty, contents of B greater than $b-a$.
   Fill A, fill B from A, empty B, empty A into B.
(2) Initial Position : A empty, contents of B less than $b-a$.
   Fill A, empty A into B.

It follows from these transformations that four operations are required to pass from a to $S^{b-2}b$, and two or four operations to pass from $S^{r+1}b$ to S^rb according as $S^{r+1}b$ is less or greater than $b-a$.

Since one operation is required initially to empty the contents of A into B, and since the last two operations of transformation (1) are unnecessary at the last application of this transformation, it follows that to find the number of operations to obtain a value S^rb we must add or subtract unity, from the total obtained as above, according as $S^{r+1}b$ is less or greater than $b-a$.

Example : $a = 5, \ b = 8$. The substitution S is given by

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 1 & 2 & 3 \end{matrix}$$

Repeated application of the substitution gives the values 8, 3, 6, 1, 4, 7, 2, 5. The value 4, for instance, is obtained by the first set of transformations in $2+4+2+4-1 = 11$ operations, and by the second set in $4+2+4-1 = 9$ operations, respectively.

It remains to prove that, provided a and b are relatively prime, then every integral number of pints, up to b, is measurable by the two vessels, *i.e.* that the numbers $b, Sb, S^2b, \dots S^{b-1}b$ are all different. If a, b have a common factor n, where $b = Kn$, we can show that the substitution S contains n cycles of K terms each. For if x is divisible by n, then both $x-a$ and $b-a+x$ are divisible by n,

whence Sx is divisible by n if x is divisible by n.   Therefore S contains a cycle, beginning with b, and containing only the K numbers less than b which are multiples of n, and ending with a.   It is easily seen that if $S(x+1) = y+1$ then $Sx = y$, and so from the cycle $a_1, a_2, \ldots a_k$, we can form the cycle $a_1 - 1, a_2 - 1, \ldots a_k - 1$, where $a_r - 1$ stands for b if $a_r = 1$.   Since $b - n$ is the first number below b which is a multiple of n, the original cycle will be reproduced after n reductions.   Thus S contains n cycles, each of K terms.   For instance, if $a = 10$, $b = 15$, the substitution S contains the five cycles, $(15, 5, 10)$, $(14, 4, 9)$, $(13, 3, 8)$, $(12, 2, 7)$, $(11, 1, 6)$.

Suppose next that a and b are relatively prime and that $b, a_1, a_2, \ldots a_k, a$, is a cycle of S ; a cycle commencing with b necessarily ends with a, since $Sa = b$.   By subtracting unity from each term of a cycle we obtain another cycle.   Let n be the least number such that $b - n$ is one of the numbers $a_1, a_2, \ldots a_k, a$, so that after n reductions the cycle $b, a_1, a_2, \ldots a_k, a$, is reproduced (with a different initial term) and the substitution S resolves into n cycles.   Then for some p, after pn reductions, the same cycle is reproduced with a as the leading term and so $b - pn = a$; but the substitution is formed of n cycles so that $b = n(k+2)$, whence it follows that $a = n(k - p + 2)$.   Thus n is a common factor of a and b, and therefore $n = 1$, and S contains the unique cycle $b, Sb, S^2b, \ldots S^{b-1}b$, consisting of all the numbers from 1 to b.   ERIC GOODSTEIN.

1504. *The transformation of* $\nabla^2 V$.

The transformation of

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

by a substitution of the form

$$x = f(u, v, w), \quad y = g(u, v, w), \quad z = h(u, v, w)$$

is usually presented with the aid of the ideas of geometry, physics, or tensors, or Gauss's integral theorem, and is often restricted to the orthogonal case.   In the following note the general case is given as a straightforward example in " change of variables ".

Let $J \neq 0$ be the determinant of the substitution and J' the reciprocal determinant, so that

$$J = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}, \qquad J' = \begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{vmatrix},$$

where $x_1 = \partial x / \partial u, \ldots, X_1 = y_2 z_3 - y_3 z_2 \ldots$.

First note the three identities of which a typical one is

$$\frac{\partial X_1}{\partial u} + \frac{\partial X_2}{\partial v} + \frac{\partial X_3}{\partial w} \equiv 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now

$$\frac{\partial V}{\partial x} \equiv \frac{\partial (V, y, z)}{\partial (x, y, z)} = \frac{1}{J} \frac{\partial (V, y, z)}{\partial (u, v, w)} = \frac{1}{J} \begin{vmatrix} V_1 & V_2 & V_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix},$$

that is,

$$\frac{\partial V}{\partial x} = \frac{1}{J} \left(X_1 \frac{\partial V}{\partial u} + X_2 \frac{\partial V}{\partial v} + X_3 \frac{\partial V}{\partial w} \right) \quad \ldots\ldots\ldots\ldots\ldots(2)$$

$$= \frac{1}{J} \left\{ \frac{\partial}{\partial u} (X_1 V) + \frac{\partial}{\partial v} (X_2 V) + \frac{\partial}{\partial w} (X_3 V) \right\},$$

by (1). Replacing V by $\partial V/\partial x$, we have

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{J} \left\{ \frac{\partial}{\partial u} \left(X_1 \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial v} \left(X_2 \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial w} \left(X_3 \frac{\partial V}{\partial x} \right) \right\}.$$

From this and two similar equations follows, by addition,

$$\nabla^2 V = \frac{1}{J} \Sigma \frac{\partial}{\partial u} \left(X_1 \frac{\partial V}{\partial x} + Y_1 \frac{\partial V}{\partial y} + Z_1 \frac{\partial V}{\partial z} \right).$$

Substituting for $\partial V/\partial x$, $\partial V/\partial y$, $\partial V/\partial z$ from (2) and two similar equations, we get

$$\nabla^2 V = \frac{1}{J} \Sigma \frac{\partial}{\partial u} \left(\frac{\Sigma X_1^2}{J} \frac{\partial V}{\partial u} + \frac{\Sigma X_1 X_2}{J} \frac{\partial V}{\partial v} + \frac{\Sigma X_1 X_3}{J} \frac{\partial V}{\partial w} \right).$$

To calculate ΣX_1^2, $\Sigma X_1 X_2$, ... put

$$A = x_1^2 + y_1^2 + z_1^2, \qquad F = x_2 x_3 + y_2 y_3 + z_2 z_3,$$
$$B = x_2^2 + y_2^2 + z_2^2, \qquad G = x_3 x_1 + y_3 y_1 + z_3 z_1,$$
$$C = x_3^2 + y_3^2 + z_3^2, \qquad H = x_1 x_2 + y_1 y_2 + z_1 z_2;$$

then we find two sets of three equations typified by

$$\Sigma X_1^2 = \Sigma \begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix}^2 = BC - F^2,$$

$$\Sigma X_2 X_3 = \Sigma \begin{vmatrix} y_3 & y_1 \\ z_3 & z_1 \end{vmatrix} \begin{Vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{Vmatrix} = GH - AF.$$

Consequently, $J \nabla^2 V =$

$$\frac{\partial}{\partial u} \frac{1}{J} \begin{vmatrix} V_1 & V_2 & V_3 \\ H & B & F \\ G & F & C \end{vmatrix} + \frac{\partial}{\partial v} \frac{1}{J} \begin{vmatrix} A & H & G \\ V_1 & V_2 & V_3 \\ G & F & C \end{vmatrix} + \frac{\partial}{\partial w} \frac{1}{J} \begin{vmatrix} A & H & G \\ H & B & F \\ V_1 & V_2 & V_3 \end{vmatrix};$$

this is the transformation desired.

Note that, by the multiplication rule for determinants,

$$J^2 = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}.$$

Cor. If the substitution is orthogonal, $F = G = H = 0$, $J^2 = ABC$, and

$$\nabla^2 V = \frac{1}{J} \left\{ \frac{\partial}{\partial u} \left(\frac{J}{A} \frac{\partial V}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{J}{B} \frac{\partial V}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{J}{C} \frac{\partial V}{\partial w} \right) \right\},$$

where J may be replaced by $|J| = (ABC)^{\frac{1}{2}}$, since the right-hand side is unaltered if we replace J by $-J$. F. BOWMAN.

1505. *Notes on Conics.* 6 : *Divagations from a problem.*

1. " A parabolic arc is to pass from P to Q, and to touch LM at O. A geometrical construction is required for the tangents PR, QR." (Note 1471, *Gazette*, XXIV, p. 215.)

Let H, K be the mid-points of OP, OQ, and let PR, QR cut LM in U, V. Then HU, KV are diameters of the parabola, and since H, K are known points, the problem is solved when the direction of the diameters is found.

The conics which pass through P and Q and touch LM at O compose a pencil ; a conic belonging to the pencil is a parabola if its intersection with the line at infinity is a pair of coincident points, and this is the case if the point of coincidence is a double point of the involution cut on that line by the pencil. Hence in general the original problem has two solutions or none, according as the involution has or has not double points. The involution is determined by the line-pairs belonging to the pencil, namely (OP, OQ) and (LM, PQ) ; in enumeration the first of these has to be reckoned twice, but since two distinct pairs of elements are sufficient to determine an involution, we need not attempt to make good the loss of one pair out of three. Of the four lines entering into our two line-pairs, three already pass through O ; it is therefore worth while to replace the range at infinity by a pencil at O : the diameters through O of the two parabolas in the pencil of conics are the double lines of the involution of which one pair is (OP, OQ) and another consists of LM and the line through O parallel to PQ. It follows that if LM cuts PQ in F, the lines required join O to the double points of the involution which has F for centre and (P, Q) for one pair.

The solution of the original problem in elementary terms is now complete. If P, Q are on opposite sides of LM there is naturally no solution ; if P, Q are on the same side of LM and if Y, Z are the two points in PQ such that $FY^2 = FZ^2 = FP \cdot FQ$, there is one parabola whose diameters are parallel to OY and one whose diameters are parallel to OZ.

The moral? That we need not keep elementary metrical properties of conics at our fingers' ends even if we may from time to time have metrical problems to solve.

2. Let us now examine the property of the parabola which we have actually discovered : *If a secant PQ of a parabola cuts the tangent at O in F and cuts the diameter through O in X, then $FX^2 = FP \cdot FQ$.* In this form the result is not familiar. What is its essence?

We notice first that F is on a second tangent and that the diameter through the point of contact of this second tangent can be substituted for the diameter through O : *If a line through a point F outside a parabola cuts the diameters through the points of contact of the tangents from F in Y, Z, and cuts the curve in P, Q, then $FY^2 = FZ^2 = FP . FQ$.* This theorem falls into two parts : (i) F *is the mid-point of* YZ ; (ii) (YZ, PQ) *is harmonic.* The points Y, Z exist whether the line cuts the curve or not, and if we replace (ii) by : (ii′) Y, Z *are conjugate for the parabola,* we have an assertion which also remains significant. And we have only to glance at a figure to recognise (i) and (ii′) as versions of elementary theorems which everybody knows.

The product $FP . FQ$ does not exist unless the line through F cuts the curve, but there is a function, the power of F along a line l through F, definable in many ways, which exists in any case and is equal to the product when the product exists. We will assume the general theorem which fits the power $\varpi (F, l)$ into our present framework :

For any conic, if Y, Z are conjugate points on a line l and if F is the mid-point of YZ, then $FY^2 = FZ^2 = \varpi (F, l)$.

Taking this theorem as known, we can recombine (i) and (ii′) : *If a line l through a point F outside a parabola cuts the diameters through the points of contact of the tangents from F in Y, Z, then $FY^2 = FZ^2 = \varpi (F, l)$.*

3. If we wish to remove the restriction that we are dealing with a parabola, we must make a direct investigation on the general conic, for the power $\varpi (F, l)$ does not survive the projection that replaces the line at infinity by an arbitrary tangent. We turn naturally to the fundamental theorem on powers, the rectangle theorem of Apollonius and Newton. Let O be a fixed point on a conic, let F be a variable point on the tangent at O, and let l be a line through F whose direction remains fixed while F varies. Then $\varpi (F, l)$, the power of F along l, bears a constant ratio to FO^2, the power of F along FO, and therefore if X is a point in l such that $FX^2 = \varpi (F, l)$, the ratio of FX^2 to FO^2 is constant and X is on one of two fixed lines through O. If l is a tangent, its point of contact satisfies the condition imposed. Hence firstly : *Let O, Q be points on a conic, let F be a point on the tangent at O, let l be the line through F parallel to the tangent at Q, and let OQ cut l in Y ; then $FY^2 = \varpi (F, l)$.* And secondly : *Let Q', Q'' be points diametrically opposite on a conic, and let an ordinate l to the diameter $Q'Q''$ cut the tangent at a point O in F and cut the lines OQ', OQ'' in Y, Z ; then $FY^2 = FZ^2 = \varpi(F, l)$.*

Curiously enough, to read for the parabola the first of these theorems as the theorem by which our original problem was solved we have to take for granted that we can say that whatever line l may be, the line at infinity is parallel to l ; if we apply the theorem only to the accessible tangent parallel to l, a further step, albeit a simple one, is necessary. We can conceal our hesitation by a modified wording which really depends on the second theorem : *Let O, Q be*

points on a conic, let F be a point on the tangent at O, let l be the line through F parallel to the tangent at Q, and let the line joining O to the point diametrically opposite to Q cut l in Z ; then $FZ^2 = \varpi(F, l)$. It seems easier to accept that in the case of a parabola " the line joining O to the point diametrically opposite to Q " can be identified accidentally as the diameter through O, than to rely uncritically on an assurance that the alternative use of parallelism is legitimate, though in truth in the present case there is no ground for misgiving.

In conclusion we introduce the second tangent through F, and we observe that since we can construct the two points Y, Z from one point of contact only, the recovery of the same two points from the other point of contact implies theorems of incidence :

Let Q', Q'' be diametrically opposite points on a conic, let O_1, O_2 be any two points of the curve, let O_1Q', O_2Q'' cut in Y and let O_1Q'', O_2Q' cut in Z ; then the points Y, Z are conjugate for the conic, the line YZ is an ordinate to the diameter $Q'Q''$, and the mid-point F of YZ is the pole of O_1O_2.

From this theorem the restriction on Q', Q'' can be removed immediately :

Let O_1, O_2 and Q_1, Q_2 be two pairs of points on a conic, let O_1Q_1, O_2Q_2 cut in Y, and let O_1Q_2, O_2Q_1 cut in Z ; then Y, Z are conjugate for the conic, and the poles of O_1O_2, Q_1Q_2 lie on the line YZ and harmonise with the points Y, Z.

The end of our wanderings is the fundamental theorem on the quadrangle-quadrilateral configuration of points and tangents.

<div align="right">E. H. N.</div>

1506. *A theorem relating to the Feuerbach point.*

ABC is a triangle. The internal bisectors of the angles A, B, C meet the opposite sides in D, E, F respectively. The circle DEF passes through the Feuerbach point of $\triangle ABC$.

Is this theorem known ?

I derived it as follows in an attempt to see what happens *when* DEF *is the triangle of reference.*

If a, b, c, R, r_1, r_2, r_3 have their usual meanings with respect to $\triangle ABC$, the sides EF, FD, DE are proportional to

$$(b+c)\surd(R+r_1), \quad (c+a)\surd(R+r_2), \quad (a+b)\surd(R+r_3)$$

respectively. Hence the areal equation of the circle DEF is

$$\Sigma(b+c)^2(R+r_1)/x = 0.$$

Again, from the powers of D, E, F with respect to the in-circle of $\triangle ABC$, the equation of the radical axis of that circle and the circle DEF is found to be

$$\Sigma(b-c)^2(s-a)^2x/(b+c)^2 = 0.$$

Similarly, the radical axis of the nine-point circle of $\triangle ABC$ and the circle DEF is

$$\Sigma(b-c)^2(s-a)x/(b+c)^2 = 0.$$

The two radical axes meet at the point

$$[(b+c)^2/(b-c)(s-a), (c+a)^2/(c-a)(s-b), (a+b)^2/(a-b)(s-c)].$$

This point lies on the circle DEF, as can be easily verified from the equation of that circle.   Hence, etc.   F. H. V. GULASEKHARAM.

1507. *The Simson lines of a cyclic quadrilateral.*

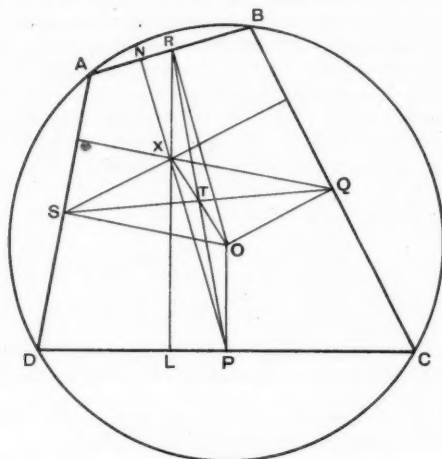If P, Q, R, S are the mid-points of the sides of the cyclic quadri-

FIG. 1.

lateral $ABCD$, the following propositions can be proved by elementary geometry :

-   (a) the four perpendiculars from P, Q, R, S to the opposite side are concurrent at a point X ;
-   (b) the Simson line of A with respect to the triangle BCD and the other three lines similarly obtained are concurrent at X ;
-   (c) if the straight lines PR, QS intersect at T, then T is the mid-point of OX.

The following investigation of (b) may be of interest to pupils who would care for further exercises in vector notation :

Let the point $(\cos \theta, \sin \theta)$ on the unit circle $x^2 + y^2 = 1$ be denoted by (θ). Then the foot of the perpendicular from (θ) to the join of (α) and (β) can easily be seen to be given by (x, y) where

$$2x = \cos \theta + \cos \alpha + \cos \beta - \cos (\alpha + \beta - \theta),$$
$$2y = \sin \theta + \sin \alpha + \sin \beta - \sin (\alpha + \beta - \theta),$$

or $2z = e^{i\theta} + e^{i\alpha} + e^{i\beta} - e^{i(\alpha+\beta-\theta)},$ ....(i)

converting our figure into an Argand diagram.

Considering the points (α), (β), (γ), (δ) therefore, and writing

$$e^{i\alpha} = a, \quad e^{i\beta} = b, \quad e^{i\gamma} = c, \quad e^{i\delta} = d,$$

the foot of the perpendicular from (α) to the join of (β) and (γ) is the vector $\frac{1}{2}(a + b + c - bc/a)$. Moving the origin from O to the point $\frac{1}{2}(a + b + c + d)$, the new vector becomes

$$-\tfrac{1}{2}\left(\frac{bc}{a} + d\right). \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

For the purpose of our work, we can finally impose a magnification of -2 on the whole figure, and thus we have the convenient result
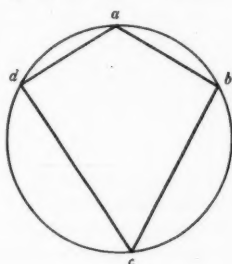
FIG. 2.

that the foot of the perpendicular from the point (α) to the join of (β) and (γ) can be represented in a similar figure with a new origin by the vector $\{(bc/a) + d\}$. We can express this by

$$_\alpha Z_{\beta\gamma} = (bc/a) + d. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

From this result several properties of the circle can be deduced in an interesting way.

The Simson line.

$$_\alpha Z_{\beta\gamma} = (bc/a) + d,$$
$$_\alpha Z_{\gamma\delta} = (cd/a) + b,$$
$$_\alpha Z_{\delta\beta} = (bd/a) + c.$$

Now vectors z_1, z_2, z_3 are collinear if

$$\frac{\lambda z_1 + \mu z_2}{\lambda + \mu} = z_3$$

where λ/μ is real. Hence $_\alpha Z_{\beta\gamma}$, $_\alpha Z_{\gamma\delta}$, $_\alpha Z_{\delta\beta}$ are collinear if the equation

$$\lambda(bc + ad) + \mu(cd + ab) = (\lambda + \mu)(bd + ac)$$

gives a *real* ratio λ/μ. Solving this equation we have

$$\frac{\lambda}{\mu} = \frac{(b - c)(a - d)}{(a - b)(c - d)};$$

· but
$$\arg \{(b-c)/(a-b)\} = \angle\, abc \quad (\text{mod } \pi),$$
$$\arg \{(a-d)/(c-d)\} = \angle\, adc \quad (\text{mod } \pi).$$

Hence λ/μ is real and $_\alpha Z_{\beta\gamma}$, $_\alpha Z_{\gamma\delta}$, $_\alpha Z_{\delta\beta}$ are collinear.

Concurrence of the four Simson lines.

Consider the Simson line defined by the join of $_\alpha Z_{\gamma\delta}$, $_\alpha Z_{\delta\beta}$. Any point K dividing the join of these points in the ratio λ/μ is given by the vector

$$\left\{ \lambda \left(b + \frac{cd}{a} \right) + \mu \left(c + \frac{bd}{a} \right) \right\} \Big/ (\lambda + \mu) \equiv Z_K,$$

so that $Z_K = C_K + iS_K$, where

$$(\lambda + \mu)\, C_K = 2 \{\lambda \cos \tfrac{1}{2}(\gamma + \delta - \alpha - \beta) + \mu \cos \tfrac{1}{2}(\beta + \delta - \alpha - \gamma)\}$$
$$\cos \tfrac{1}{2}(\beta + \gamma + \delta - \alpha),$$

$$(\lambda + \mu)\, S_K = 2 \{\lambda \cos \tfrac{1}{2}(\gamma + \delta - \alpha - \beta) + \mu \cos \tfrac{1}{2}(\beta + \delta - \alpha - \gamma)\}$$
$$\sin \tfrac{1}{2}(\beta + \gamma + \delta - \alpha),$$

or
$$(\lambda + \mu)\, Z_K = 2 \{\lambda \cos \tfrac{1}{2}(\gamma + \delta - \alpha - \beta) + \mu \cos \tfrac{1}{2}(\beta + \delta - \alpha - \gamma)\}$$
$$\exp \tfrac{1}{2} i (\beta + \gamma + \delta - \alpha).$$

Thus a *real* ratio for λ/μ is given for $Z_K = 0$. Hence the join of $_\alpha Z_{\gamma\delta}$, $_\alpha Z_{\delta\beta}$ passes through the origin, that is, the point $\tfrac{1}{2}(a + b + c + d)$ in the original figure. Similarly the other Simson lines pass through $\tfrac{1}{2}(a + b + c + d)$ which is easily shown to be the point X in the figure.

R. F. CYSTER.

1508. *A note on Simson's pedal line.*

If D be a point on the circumcircle of a triangle ABC, the following two theorems on the Simson line (S-line) of D with respect to ABC are found in school textbooks on geometry:

Theorem I. If DX is drawn perpendicular to BC to meet the circle again at X, then the S-line of D is parallel to AX.

Theorem II. If H be the orthocentre of $\triangle ABC$, the S-line of D bisects HD.

The following theorem may be added:

Theorem III. If ABC, DEF be two triangles inscribed in a circle *such that the S-line of D with respect to ABC is perpendicular to EF*, then the S-lines of D, E, F with respect to $\triangle ABC$ and the S-lines of A, B, C with respect to $\triangle DEF$ meet in a point W, which is the middle point of HH', where H and H' are the orthocentre of the triangles ABC and DEF.

Proof. We can prove easily, by using Theorem I, that the S-lines of E, F with respect to ABC, and the S-lines of A, B, C with respect to DEF are at right angles to FD, DE, BC, CA, AB respectively.

Let D', E', F', A', B', C' be the middle points of HD, HE, HF, $H'A$, $H'B$, $H'C$ respectively.

Now the S-line of D with respect to ABC passes through D' (Theorem II) and is at right angles to EF and therefore parallel to DH' and hence to $D'W$.

Hence and similarly the S-lines of D, E, F, A, B, C are $D'W$, $E'W$, $F'W$, $A'W$, $B'W$, $C'W$ respectively. Hence the theorem.

Theorem IV (The orthopole of EF). If L, M, N be the projections of A, B, C respectively on a straight line which cuts the circle ABC at E and F, then the perpendiculars from L, M, N on BC, CA, AB respectively meet in a point, called *the orthopole of EF*, which point is also the intersection of the S-lines of E and F with respect to $\triangle ABC$.

Proof. Let AL meet the circle again at X, and let XD perpendicular to BC meet the circle again at D. Then, by Theorem I, the S-line of D with respect to ABC is parallel to AX and therefore perpendicular to BC. Hence, by Theorem III, the S-lines of D, E, F with respect to ABC and the S-lines of A, B, C with respect to DEF meet in a point W.

Again, the S-line of A with respect to DEF passes through W and L, and is parallel to DX and hence perpendicular to BC.

Hence and similarly WL, WM, WN are at right angles to BC, CA, AB respectively.

Hence the theorem.

Theorem V. *A necessary and sufficient condition* that two triangles ABC, DEF inscribed in a circle may be so related that the S-lines of the vertices of either triangle with respect to the other meet in a point is *that the S-line of D with respect to ABC is at right angles to EF*.

Proof. Theorem III proves that the condition is *sufficient*.

To prove that the condition is *necessary* :

Let ABC, DEF be two triangles inscribed in a circle such that the S-lines of D, E, F with respect to triangle ABC meet in a point W.

Let DX at right angles to BC meet the circle again at X, and let EF' at right angles to AX meet the circle again at F'. Then, by Theorem III, the S-lines of D, E, F' meet in a point, *which must be* W. Then, by Theorem IV, W is the orthopole of the lines ED, EF, EF'. Hence if A_1, A_2, A_2' respectively be the projection of A on these lines, then $WA_1A_2A_2'$ is a straight line at right angles to BC, which is impossible, since A_1, A_2, A_2' must lie on the circle of which AE is a diameter.

Hence $A_2' \equiv A_2$, and therefore $EF'' \equiv EF$.

Hence the S-line of D with respect to $\triangle ABC$ is at right angles to EF.

Theorems III and V may now be re-stated as follows :

If ABC, DEF be two triangles inscribed in a circle such that the S-lines of D, E, F with respect to ABC meet in a point W, then the S-lines of A, B, C with respect to DEF meet in the same point W, which is the middle point of the line joining the orthocentres of the two triangles.

Equivalent conditions :

(1) An *analytical* equivalent of the necessary and sufficient condition of Theorem V is

$$(\alpha + \beta + \gamma) - (\theta + \phi + \psi) = 2n\pi,$$

where α, β, γ, θ, ϕ, ψ are the angles which OA, OB, OC, OD, OE, OF make with any straight line, O being the centre of the circle ABC.

(2) An equivalent *geometrical condition* is the following :

There exists a unique point w on the circle such that the projec-- tions of w on the sides of the triangle are collinear.

This is again equivalent to the condition :

A parabola can be drawn to touch the sides of the two triangles.

The following may be suggested as exercises in elementary geometry :

If ABC, DEF are two triangles inscribed in a circle *such that the S-line of D with respect to ABC is parallel to EF*, then the S-lines of E and F with respect to ABC are *parallel to FD, DE* respectively ; if the three S-lines form a triangle PQR, then the triangles PQR and ABC are such that the sides of either are S-lines with respect to the other ; the circumcentre of $\triangle PQR$ is at the middle point of the line joining the orthocentres of the triangles ABC and DEF.

I have not seen anywhere a purely geometrical treatment of the above properties of pedal lines.   Hence this note.

<div align="right">F. H. V. Gulasekharam.</div>

1509. *A useful (and curiously good) approximation.*

Making no allowance for investment value, an annual salary of £x is approximately equivalent to a weekly wage of $\frac{2}{5}x$ shillings less $\frac{1}{5}x$ pence.

Thus   £100 per annum

$$\simeq 40\text{s}. - 20\text{d}.$$

$$\simeq £1 \ 18\text{s}. \ 4\text{d}. \ \text{per week}.$$

The error, taking a year as 365 days, is only $0 \cdot 06\%$ and only $0 \cdot 01\%$ if leap year is allowed for.

To cope with extreme longevity, let me add that the error is $0 \cdot 007\%$ when the century non-leap-years are counted.   H. V. S.

1510. *Why all this fuss?   (Instances from scripts.)*

(i) Correct answer

$$= 3 \log_e 7 = 5 \cdot 837 \ (7).$$

But why all this fuss?

$$3 \log_e 7 = \log_e 21 = (\log_{10} 21)/(\log_{10} e)$$

$$= (1 \cdot 3222)/(0 \cdot 4343).$$

Now

1·3222	0·1212
0·4343	0·6378
	0·3834

(where the figure $0 \cdot 3834$ should have been $0 \cdot 4834$) ; then $0 \cdot 3834$ was looked out in the *logarithm* table and the result $5 \cdot 837$ obtained.

(ii) To evaluate

$$\sqrt{(3\cdot142 \times 0\cdot8437 \times 0\cdot2143)}.$$

3·142	0·4972
0·8437	$\bar{1}$·9216
0·2143	$\bar{1}$·3310
	$\bar{1}$·7498

Then the answer was given as

$$0\cdot7498.$$

But, the correct result is

$$\text{antilog } (\tfrac{1}{2} \times \bar{1}\cdot7498)$$
$$= \text{antilog } (\bar{1}\cdot8749)$$
$$= 0\cdot7498.$$

A further coincidence is that the same four figures (7498) occur in the same cyclic order (8749) in the two " final " logarithms.

1344. I have a vague recollection of when I was at school being taught something called " permutations and combinations ", and I have some idea that that meant joining up something with everything else in turn and always getting the wrong answer.—General Sir Charles Harrington, in *The Sphere*, 16th December, 1939, p. 348.

1345. It was all very nice being faced by an algebraic problem containing many unknown quantities and having only one biquadratic equation with which to solve it.—Eric Ambler, *The Mask of Dimitrios*, p. 146. [Per Dr. G. J. Lidstone.]

1346. Sir James Jeans tells us the Deity is a great Mathematician, a magnified and non-material Jeans. I find this rather disquieting ; a mathematical deity would certainly plough me.—Dean Inge, in *The Evening Standard*, 1st January, 1940. [Per Mr. L. W. H. Hull.]

1347. I am no mathematician, but I believe a straight line is, or was, the shortest way between two points, but I am told that curved movements are preferable to straight lines—whether on grounds of aesthetics or comfort, I am not sure.—Earl Baldwin. [Per Mr. A. F. Mackenzie.]

1348. SAMUEL JOHNSON ON PROBABILITY.
The inhabitants of Sky, and of the other islands, which I have seen, are commonly of the middle stature, with fewer among them very tall or very short, than are seen in England ; or perhaps, as their numbers are small, the chances of any deviation from the common measure are necessarily few.— Samuel Johnson, *Journey to the Western Islands of Scotland*, par. 308. [Per Mr. A. F. Mackenzie.]

1349. Lord Justice Slesser said he supposed " the cow's case " was that cows had strayed about the road from time immemorial. The cow's drover had said that nobody could read the mind of a cow and it was a fact that cows did not proceed in straight lines under Newton's law.—Report of a court case in *The Express and Echo*, 9th October, 1939. [Per Mr. A. F. Mackenzie.]

REVIEWS.

Geomagnetism. By S. CHAPMAN and J. BARTELS. 2 vols. Pp. xxviii, 1049. 63s. 1940. (Oxford University Press)

Everyone interested in the subject of the earth's magnetism and solar-terrestrial relationships will give unstinted welcome to this comprehensive work, *Geomagnetism*, which Professor Sydney Chapman has written in collaboration with Dr. J. Bartels. The increasing importance of geomagnetism, which includes aspects of solar physics, upper air meteorology and radio transmission, called for a modern textbook of its own.* Hitherto, the student and research worker has had to delve for himself into a variety of scientific journals (of which one only is specifically devoted to terrestrial magnetism) to cover the widening field of this branch of science. How wide that field has become is indicated by the reference to more than 1,000 papers in the present work.

The general plan of the two volumes is as follows. In Part I is given an account of the observed facts of geomagnetism and the methods by which they are found and recorded. Chapters on solar phenomena, earth currents, magnetism and geology, the upper atmosphere, cosmic rays and the results of radio soundings of the ionosphere are included. There is an illuminating account of the aurora. In Part II it is shown how the great array of data thus assembled is analysed and synthesised. This section includes a chapter dealing with the statistical basis for the treatment of periodicities in geomagnetism. Special care is needed, for, as the authors point out, many serious misunderstandings that have arisen are due to the fact that methods suitable in biometrical studies have been inappropriately applied to geomagnetism. Tests for the reality of periodicities in geophysical time series " are often based on assumptions as to randomness which ignore the important characteristic of ' conservation ' which is a notable feature in many kinds of geophysical data ". Another chapter deals with spherical harmonic analysis which mathematically provides the only means of separating the interior and exterior parts of the earth's magnetic field. Part III contains the discussion of the physical causes and mechanism of the phenomena under review. It includes the mathematically developed theories of magnetic storms and of the solar and lunar daily magnetic variations : the geometry of solar streams of corpuscles and " corpuscular " solar eclipses. To all these subjects Professor Chapman and his associates have made important contributions.

Judged from laboratory standards, progress in geomagnetism is very slow. The reason is easily appreciated from the following extract. " In general, a function $u = f(x, y, z \ldots)$ is experimentally evaluated by changing only one variable at a time, etc. But this simplification is only rarely possible in geophysics. Consider the magnetic field at a given station. It depends on the situation of the station on the earth, on the underground structure, and, above all, it is a very complicated function of the time ; it depends on the year, the season, on solar and lunar time, and on the solar and magnetic activity at the time of observation and some time before. It is not possible for us to modify the course of these variables. In order to separate their influences, it is necessary to have data in which the variables appear in different combinations ; we must also have observations from many stations distributed over the globe in order to find the influence of the geographical

* The publication of the present work was preceded some months ago by an American work entitled *Terrestrial Magnetism and Electricity* edited by J. A. Fleming.

coordinates. In order to separate the various influences depending on the time, continuous registration over long series of years is of great importance."

This need for very extensive data may be instanced in the case of the very small regular periodic change in the earth's magnetic field depending on the lunar hour angle. In Chapman's discussion of the Greenwich observations, which give the most reliable determination of this phenomenon, no fewer than sixty-three years' data were used. Radio soundings of the ionosphere—by methods due to Appleton in particular—have in recent years been applied with marked success to certain problems of geophysics and have yielded significant results within a relatively short period of observation. Thus it has been inferred from direct reflection observation of radio waves, carried out over a period of about a year, that there exists a lunar semi-diurnal variation, tidal in character, of the equivalent height of the E region of the ionosphere. Another instance, showing the use of new methods, concerns the height of the inferred overhead current systems giving rise to the daily solar variation of the magnetic elements. Magnetic data alone leave the height conjectural. A valuable clue is provided, however, by radio soundings of the ionosphere interpreted in conjunction with the observation of bright solar eruptions and small associated terrestrial magnetic effects. The general height of the current system suggested by these data is just below the base of the E layer : say, of the order of 90 kms. The systematic observation of the solar eruptions themselves is due to a novel form of solar spectroscope brought generally into use about ten years ago.

With these brief remarks, we wish to call attention to this new treatise on the earth's magnetism, to which every worker in the field of geophysics will turn frequently for information and guidance. The general arrangement and printing of the two volumes, the historical notes, extensive bibliography, and the set of tables of magnetic and solar data give completeness and distinction to an important work.   H. N.

An Introduction to the Kinetic Theory of Gases. By Sir JAMES JEANS. Pp. 311. 15s. 1940. (Cambridge)

This book can be described as a student's edition of the author's *Dynamical Theory of Gases* ; it is written from the same point of view, and in places uses the same words. It is written, however, with the needs of the student of physics and physical chemistry in mind, and those parts of the old book of which the interest was mainly mathematical have been discarded. This does not mean that the book contains no serious mathematical discussion ; the discussion in particular of the distribution law is quite detailed ; but in the main the mathematics is concerned with the discussion of particular phenomena rather than with the discussion of fundamentals.

There are chapters on the gas laws, on Maxwell's distribution laws, on diffusion, viscosity and the conduction of heat, and on the specific heats of gases, excluding such parts of the theory as depend on the quantum theory. The chapters on viscosity and kindred subjects have been brought up to date by a discussion of Chapman's work and of calculations using modern forms of the interatomic forces. There is throughout a careful comparison between theory and recent experimental results. This part of the book would, however, have been more complete if some mention had been made of experiments on the mobility of ions ; this seems a curious omission from a book at least half of which is devoted to the discussion of phenomena dependent on the mean free path.

A large part of the book seems to us very suitable for an Honours student of Physics ; as is to be expected from Sir James Jeans, physical principles are

discussed in a most lucid way and with the help of useful analogies, as, for instance, in his comparison between the distribution of molecular velocities and of rifle shots on a target. For students, however, we doubt whether it is wise to give a complete account of the kinetic theory of gases while making no mention of the quantum theory. In 1940 the quantum theory cannot be regarded as new or revolutionary; the student will certainly have heard about it before he reads a book such as this, and some of its more elementary applications, such as the drop in the specific heat of hydrogen at low temperatures, are surely better included in a book on kinetic theory than in a book specifically devoted to quantum phenomena. This criticism does not, however, apply to the parts which deal with the mean free path, which can be unreservedly recommended.   N. F. M.

Bibliography of mathematical works printed in America through 1850. By L. C. KARPINSKI. Pp. xxvi, 697. 6 dollars. 1940. (University of Michigan Press; Humphrey Milford)

To examine and criticise at all adequately this solid result of Professor Karpinski's long-continued bibliographical researches into the history of American mathematics would entail investigations for which not only is the present time unsuited but for which the material available in this country is probably insufficient. The author has had access to many of the well-known American collections, including the magnificent Plympton library, so that it is not surprising to find that the volume leaves the impression that here is the result of a task boldly conceived and efficiently carried out; later work may lead to revision of detail, but future workers in this domain will find a secure authority on which to build.

The reader should be warned that the word " through " in the title implies " from the beginnings up to the end of ", and informed that the dry bones of listing are here and there clothed by excellent engravings of title-pages of rare or important volumes. Even light relief is supplied by entries which exhibit a pleasing optimism and light-heartedness on the part of authors or publishers in the petty detail of numbering editions.

More serious topics of investigation are suggested by some entries. For instance, the popularity of the Brewster-Carlyle translation of Legendre's *Géométrie* prompts the present writer to ask if there exists a really thorough and detailed account of the influence of the great French mathematicians of the period 1770-1850 on American mathematics; there are, of course, several short studies on this matter, with some indication of the parts played by Franklin and Jefferson, but a final account is not known to me. The point is one of interest and importance as regards the development of American mathematics, and Professor Karpinski's book provides, indirectly, much relevant information.

It is always a pleasure to recognise, as we can here, a job done once and for all.   T. A. A. B.

1350. With the generally accepted fact that the density of the earth was 40,000,000 ft. from pole to pole, they were not concerned. . . .

But there were large areas in the world where soil exhaustion was very prevalent, and in the last few years it had reached astronomical proportions.— Dr. G. V. Jacks; Report in *The Times* of the Conference of the Men of the Trees. [Per Mr. A. S. Ramsey.]